# What simulations tell us about granular suspension rheology ?

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Lecture day at IUSTI. Tuesday, October 22nd.

# Outline

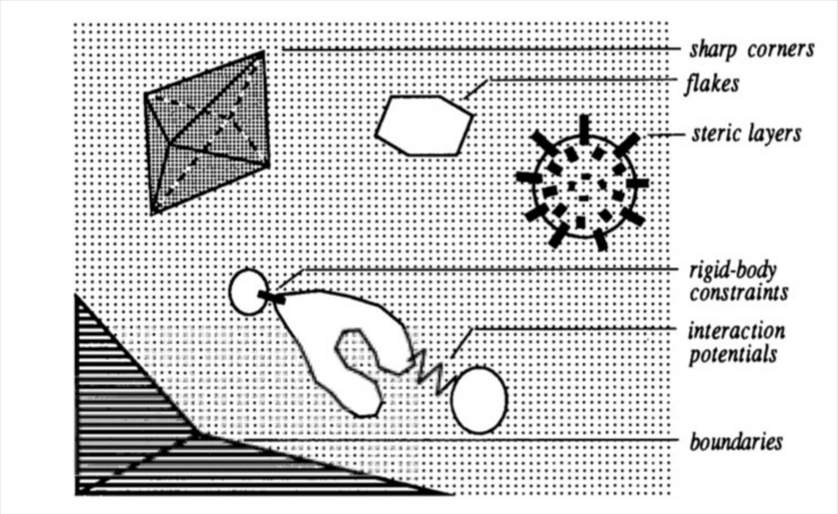
A bit of hydrodynamics

- Hydrodynamic interaction matrices
- Mean suspension stress
- Some numerical methods

Contacting particles

- Historical perspective
- Roughness and friction
- Shear reversal
- Shear thickening as a lubricated-to-frictional transition
- Shear thinning in frictional non-Brownian suspensions

# Scope



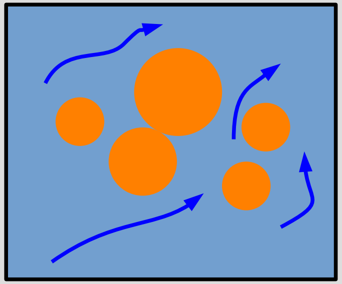

S. Kim and S.J. Karrila. Microhydrodynamics: principles and selected applications . Dover (2005).

Suspensions / emulsions / foams

Particles
- Fluid, solid, visco-elastic
- Size, shape, surface properties and interactions
- Buoyancy

Suspending fluid
- Non-Newtonian

Brownian motion



Here
- "rigid" particles = no vesicle, droplet ...
- Short range ("contact") interactions (solid-solid, grafted polymer chains, double-layer ...)
- Newtonian liquid
- No Brownian motion
- No particle nor fluid inertia

# What's left?

Boundary conditions

$\partial\mathcal{D}_p$

**Hydrodynamics**

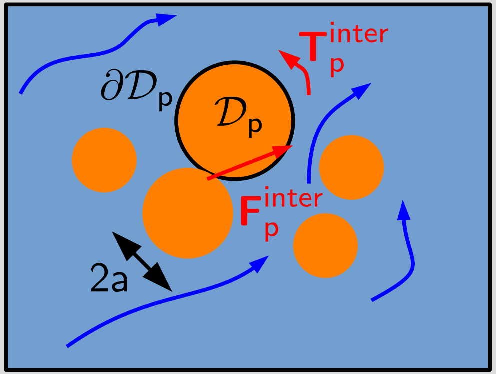
- Flow $\mathbf{u}(\mathbf{x}), P(\mathbf{x})$

Momentum balance equation

- Hydro forces and torques

$$\left(\mathbf{F}_p^h, \quad \mathbf{T}_p^h\right) = \mathcal{F}(\mathbf{u}, P)$$

- Gravity force




$\partial\mathcal{D}$

**Particles**

- Rigid $\mathbf{U}_p, \mathbf{\Omega}_p$

$$\mathbf{x} \in \mathcal{D}_p \quad \mathbf{u}(\mathbf{x}) = \mathbf{U}_p + \mathbf{\Omega}_p \times (\mathbf{x} - \mathbf{x}_p)$$

- Direct Interaction forces / torques

$$\mathbf{F}_p^{inter}, \quad \mathbf{T}_p^{inter} \qquad \text{modelling}$$

- Gravity force

**Strongly coupled equations**

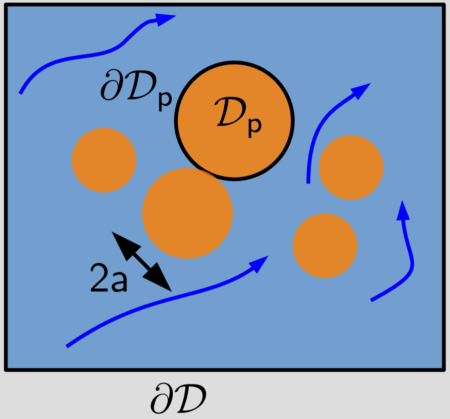
Particle linear / angular momentum balance equations

$$\mathbf{F}_p^h + \mathbf{F}_p^{inter} + (\rho_p - \rho_f)V_p\mathbf{g} = \cancel{M_p \frac{d\mathbf{U}_p}{dt}}$$

$$\mathbf{T}_p^h + \mathbf{T}_p^{inter} = \cancel{I_p \frac{d\mathbf{\Omega}_p}{dt}}$$

# Governing equations : hydrodynamics

S. Kim and S.J. Karrila. Microhydrodynamics: principles and selected applications . Dover (2005).




Newtonian liquid

$$\boldsymbol{\sigma}_m = -P_m \mathbf{I} + \eta \left( \boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^{\dagger} \right)$$

$$P_m = P - (P_0 - \rho g z)$$

Momentum balance equation

$$\boxed{\begin{array}{c} \eta \Delta \mathbf{u} - \boldsymbol{\nabla} P_m = \cancel{\rho \frac{d\mathbf{u}}{dt}} \\ \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \end{array}}$$

$$R_e = \frac{\rho U a}{\eta} \ll 1$$

$$St = \frac{a^2 \rho}{T \eta} \ll 1$$

Stokes equations

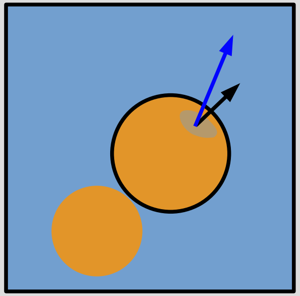
Boundary conditions

- Particles $\quad \mathbf{x} \in \mathcal{D}_p \quad \mathbf{u}(\mathbf{x}) = \mathbf{U}_p + \boldsymbol{\Omega}_p \times (\mathbf{x} - \mathbf{x}_p)$
- External boundaries $\quad \mathbf{x} \in \partial\mathcal{D} \quad \mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x})$

Forces and torques

$$\mathbf{F}_p^h = \iint_{\partial\mathcal{D}_p} \boldsymbol{\sigma}_m \cdot \mathbf{n} \, dS$$

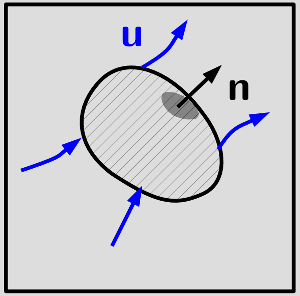
$$\mathbf{T}_p^h = \iint_{\partial\mathcal{D}_p} (\mathbf{x} - \mathbf{x}_p) \times \boldsymbol{\sigma}_m \cdot \mathbf{n} \, dS$$



# Stokes equations : some fundamental properties

$$\boxed{\begin{aligned} \eta\Delta\mathbf{u} - \boldsymbol{\nabla} P_m &= 0 \\ \boldsymbol{\nabla}\cdot\mathbf{u} &= 0 \end{aligned}}$$

- No inertia $\Longleftrightarrow$




Any fluid volume

$\mathbf{F}^h = 0$

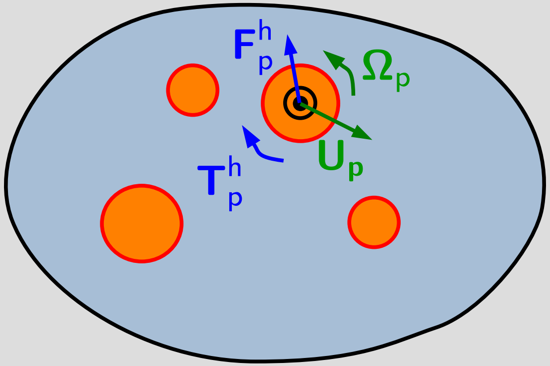
- Flow instantaneously driven by the B.C.

- Linear equations for $(\mathbf{u}(x), P_m(x))$ $\Longrightarrow$ Linear relation Hydrodynamic forces / Velocities

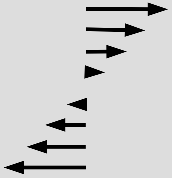
N particles in a **linear** flow $\quad \mathbf{x} \in \partial\mathcal{D} \quad \mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty(\mathbf{x}) = \mathbf{U}_0 + \boldsymbol{\Omega}_\infty \times \mathbf{x} + \mathbf{E}_\infty \cdot \mathbf{x}$

Simple shear
Extensional flow ...

$\partial\mathcal{D}$ $\quad \mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty(\mathbf{x})$




$$\mathcal{F} = (\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{T}_1, \mathbf{T}_2, \ldots)$$
$$\mathcal{U} = (\mathbf{U}_1, \mathbf{U}_2, \ldots, \boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \ldots)$$
$$\mathcal{U}_\infty = (\mathbf{u}_\infty(\mathbf{x}_1), \mathbf{u}_\infty(\mathbf{x}_2), \ldots, \boldsymbol{\Omega}_\infty, \boldsymbol{\Omega}_\infty, \ldots)$$



$$\boxed{\mathcal{F} = -\mathcal{R}_{\mathcal{F}\mathcal{U}} \cdot (\mathcal{U} - \mathcal{U}_\infty) + \mathcal{R}_{\mathcal{F}\mathrm{E}} : \mathbf{E}_\infty}$$

$\mathcal{R}_{\mathrm{XY}}$ Resistance tensors

$(\mathcal{R}_{\mathcal{F}\mathcal{U}}, \mathcal{R}_{\mathcal{F}\mathrm{E}}) = \mathrm{f}(\mathbf{x}_1, \mathbf{x}_2, \ldots)$ Not to be computed exactly (in general)

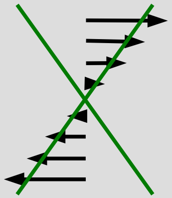
# Two-particle grand resistance matrix

Jeffrey, D. J., & Onishi, Y. (1984). *Journal of Fluid Mechanics*
Jeffrey, D. J. (1992). *Physics of Fluids A: Fluid Dynamics*
Jeffrey, D., Morris, J., & Brady, J. (1993). *Phys. Fluids A*.

- Known exactly (= analytical / semi-analytical) for any particle distance

Stresslet : later

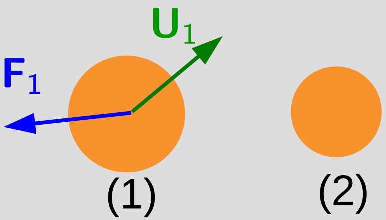
$$
\begin{pmatrix} \boxed{\mathbf{F}_1} \\ \mathbf{F}_2 \\ \mathbf{T}_1 \\ \boxed{\mathbf{T}_2} \\ \mathbf{S}_1 \\ \mathbf{S}_2 \end{pmatrix}
= -\eta
\underbrace{\begin{pmatrix}
\boxed{\mathbf{A}_{11}} & \mathbf{A}_{12} & \tilde{\mathbf{B}}_{11} & \tilde{\mathbf{B}}_{12} & \tilde{\mathbf{G}}_{11} & \tilde{\mathbf{G}}_{12} \\
\mathbf{A}_{21} & \mathbf{A}_{22} & \tilde{\mathbf{B}}_{21} & \tilde{\mathbf{B}}_{22} & \tilde{\mathbf{G}}_{21} & \tilde{\mathbf{G}}_{22} \\
\mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{C}_{11} & \mathbf{C}_{12} & \tilde{\mathbf{H}}_{11} & \tilde{\mathbf{H}}_{12} \\
\boxed{\mathbf{B}_{21}} & \mathbf{B}_{22} & \mathbf{C}_{21} & \mathbf{C}_{22} & \tilde{\mathbf{H}}_{21} & \tilde{\mathbf{H}}_{22} \\
\mathbf{G}_{11} & \mathbf{G}_{12} & \mathbf{H}_{11} & \mathbf{H}_{12} & \mathbf{M}_{11} & \mathbf{M}_{12} \\
\mathbf{G}_{21} & \mathbf{G}_{22} & \mathbf{H}_{21} & \mathbf{H}_{22} & \mathbf{M}_{21} & \mathbf{M}_{22}
\end{pmatrix}}_{\substack{\text{Tensors of order 2 to 4} \\ = f(\mathbf{x}_1, \mathbf{x}_2, \ldots)}}
\cdot
\begin{pmatrix}
\boxed{\mathbf{U}_1} - \cancel{\mathbf{u}_\infty(\mathbf{x}_1)} \\
\cancel{\mathbf{U}_2 - \mathbf{u}_\infty(\mathbf{x}_2)} \\
\cancel{\boldsymbol{\Omega}_1 - \boldsymbol{\Omega}_\infty} \\
\cancel{\boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_\infty} \\
\cancel{-\mathbf{E}_\infty} \\
\cancel{-\mathbf{E}_\infty}
\end{pmatrix}
$$



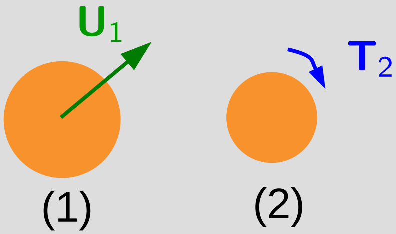
Example

$$\mathbf{u}_\infty = 0$$
$$\mathbf{U}_2 = 0$$
$$\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_2 = 0$$

$$\mathbf{F}_1 = -\eta \mathbf{A}_{11} \cdot \mathbf{U}_1$$

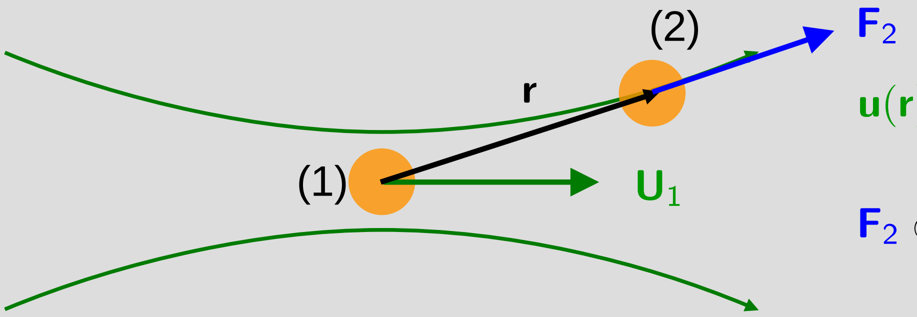



$$\mathbf{T}_2 = -\eta \mathbf{B}_{21} \cdot \mathbf{U}_1$$



# Hydrodynamic interactions : both long and short range

- r >> $a_1$, $a_2$

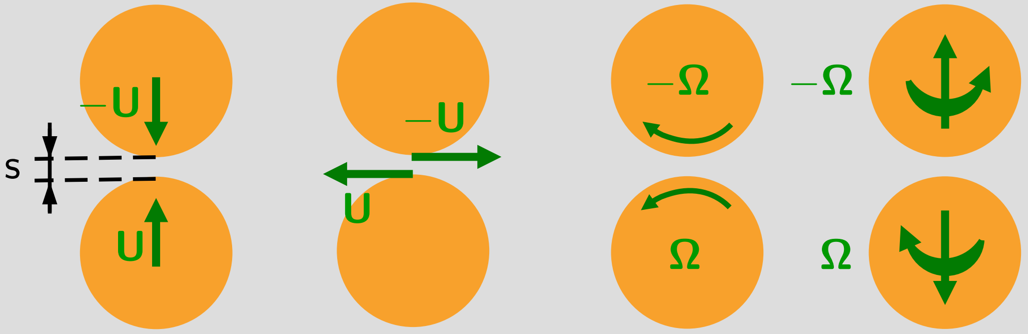



$$\mathbf{u}(\mathbf{r}) \propto \frac{a_1}{r}\mathbf{U}_1$$

$$\mathbf{F}_2 \propto 6\pi\eta a_2 \frac{a_1}{r}\mathbf{U}_1$$

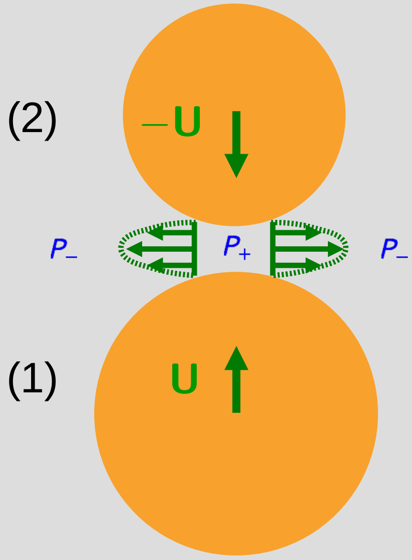
- Very long range (1/r)
- Primary role when external forces are considered (sedimentation ...)

- s = r-($a_1$+$a_2$) << $a_1$, $a_2$ Lubrication approximation




Squeeze Shear Pump Twist

$$\xi = \frac{s}{(a_1 + a_2)/2} << 1$$




- Squeeze

$$\boxed{\mathbf{F}_1, \mathbf{F}_2 \propto 1/\xi}$$

- Other terms

$$\boxed{\propto \log(\xi)}$$

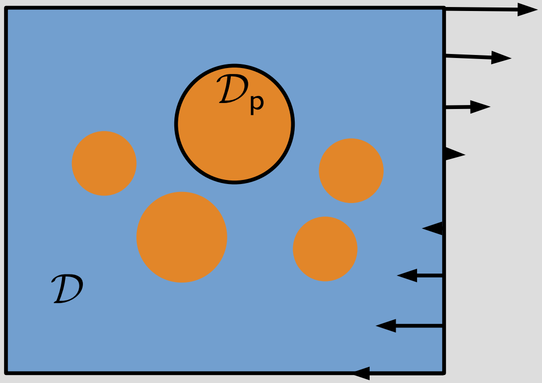
or smaller

Ball & Melrose (1997) Physica A.

# Mean suspension stress

Batchelor, G. K. (1970). Journal of fluid mechanics, 41(3), 545-570.

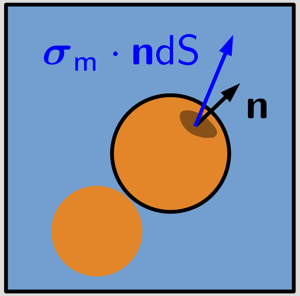
$\partial\mathcal{D} \quad \mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty(\mathbf{x})$




Mean stress $\quad \Sigma_{\mathrm{ij}} = \dfrac{1}{\mathrm{V}} \displaystyle\iiint_{\mathcal{D}} \sigma_{\mathrm{ij}} \mathrm{dV}$

$$\boxed{\Sigma_{\mathrm{ij}} = -(1-\phi)\langle \mathrm{P} \rangle_\mathrm{f} \delta_{\mathrm{ij}} + \eta \left( \frac{\partial \mathrm{u_i}}{\partial \mathrm{x_j}} + \frac{\partial \mathrm{u_j}}{\partial \mathrm{x_i}} \right) + \frac{1}{\mathrm{V}} \sum_{\mathrm{p}=1}^{\mathrm{N}} \mathrm{D}^{\mathrm{p}}_{\mathrm{ij}}}$$

Mean fluid pressure | Mean fluid deformation rate | Particle contribution

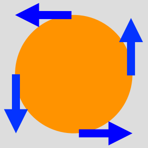



$$\mathbf{D}_\mathrm{p} = \iint_{\partial\mathcal{D}_\mathrm{p}} \boldsymbol{\sigma}_\mathrm{m} \cdot \mathbf{n} \otimes (\mathbf{x} - \mathbf{x}_\mathrm{p})\ \mathrm{dS} = \mathcal{T}_\mathrm{p} + \mathbf{S}_\mathrm{p}$$

Second moment of the force distribution = **Force dipole**

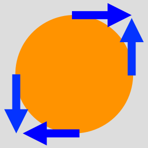
$\mathcal{T}_\mathrm{p}$ Antisymmetric part

$\Longleftrightarrow$ torque



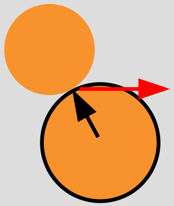
Induces particle rotation

$\mathbf{S}_\mathrm{p}$ Symmetric part = stresslet



Induces particle deformation

Stress = hydro contribution + point contact forces



$$\boxed{\begin{aligned} \mathcal{T}_\mathrm{p} &= \mathcal{T}^\mathrm{c}_\mathrm{p} + \mathcal{T}^\mathrm{h}_\mathrm{p} = 0 \\ \mathbf{S}_\mathrm{p} &= \mathbf{S}^\mathrm{c}_\mathrm{p} + \mathbf{S}^\mathrm{h}_\mathrm{p} \end{aligned}}$$

No inertia

$$\mathbf{S}^\mathrm{c}_\mathrm{p} = \frac{1}{2}\left(\Delta\mathbf{r} \otimes \mathbf{F}^\mathrm{c} + \mathbf{F}^\mathrm{c} \otimes \Delta\mathbf{r}\right)$$

<u>N.B.</u> Another splitting hydro / contact is possible (next slide)

# Hydrodynamic and contact contributions

Two different ways of separating contact and hydrodynamic contributions :

- The « Stokesian dynamics » way

Phung et al. (1996). . JFM, 313, 181-207.
Banchio and Brady, J. Chem. Phys. 118, 10323 (2003).
Mari et al. (2015). Phys. Rev. E, 91(5), 052302.

$$\boldsymbol{\mathcal{S}} = \underbrace{\overbrace{\left[-\mathcal{R}_{\mathcal{SU}} \cdot \mathcal{R}_{\mathcal{FU}}^{-1} \cdot \mathcal{R}_{\mathcal{F}\mathsf{E}} + \mathcal{R}_{\mathcal{S}\mathsf{E}}\right] : \mathbf{E}_\infty \;-\; \mathcal{R}_{\mathcal{SU}} \cdot \mathcal{R}_{\mathcal{FU}}^{-1}}^{\text{Hydro. in nature (viscous stress)}}}_{\ldots} \cdot \mathbf{F}^{\mathsf{c}} + \overbrace{\frac{1}{2}\left(\Delta\mathbf{r} \otimes \mathbf{F}^{\mathsf{c}} + \mathbf{F}^{\mathsf{c}} \otimes \Delta\mathbf{r}\right)}$$

$$\underbrace{\left[-\mathcal{R}_{\mathcal{SU}} \cdot \mathcal{R}_{\mathcal{FU}}^{-1} \cdot \mathcal{R}_{\mathcal{F}\mathsf{E}} + \mathcal{R}_{\mathcal{S}\mathsf{E}}\right] : \mathbf{E}_\infty}_{\substack{\text{Produced by the external shear w/o forces} \\ \text{= hydro}}} \underbrace{- \mathcal{R}_{\mathcal{SU}} \cdot \mathcal{R}_{\mathcal{FU}}^{-1} \cdot \mathbf{F}^{\mathsf{c}} + \frac{1}{2}\left(\Delta\mathbf{r} \otimes \mathbf{F}^{\mathsf{c}} + \mathbf{F}^{\mathsf{c}} \otimes \Delta\mathbf{r}\right)}_{\substack{\text{Produced by the forces w/o external shear} \\ \text{= contact}}}$$

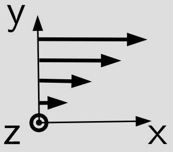
Physical meaning in shear reversal (Peters et al. 2016 J.O.R) and shear rotation (Blanc et al. 2023 PRL)

More about this on Wednesday 11h15 : Romain Mari

- The « direct » way

$$\boxed{\boldsymbol{\Sigma} = -(1-\phi)\langle \mathsf{P}\rangle_{\mathsf{f}}\boldsymbol{\delta} + \eta\left(\boldsymbol{\nabla}\otimes\mathbf{u} + \boldsymbol{\nabla}\otimes\mathbf{u}^{\dagger}\right) + \underbrace{\frac{1}{\mathsf{V}}\sum_{\mathsf{p}=1}^{\mathsf{N}}\mathbf{S}_{\mathsf{p}}^{\mathsf{h}}}_{\boldsymbol{\Sigma}^{\mathsf{h}}} + \underbrace{\frac{1}{\mathsf{V}}\sum_{\mathsf{p}=1}^{\mathsf{N}}\mathbf{S}_{\mathsf{p}}^{\mathsf{c}}}_{\boldsymbol{\Sigma}^{\mathsf{c}}}}$$

- Shear flow




Reduced Viscosity

$$\boxed{\eta_{\mathsf{r}} = \frac{\Sigma_{\mathsf{xy}}}{\eta\dot{\gamma}} = 1 + \frac{\Sigma_{\mathsf{xy}}^{\mathsf{h}} + \Sigma_{\mathsf{xy}}^{\mathsf{c}}}{\eta\dot{\gamma}} = \eta^{\mathsf{h}} + \eta^{\mathsf{c}}}$$

Normal stresses

$$\boxed{\mathsf{N}_1 = \Sigma_{\mathsf{xx}} - \Sigma_{\mathsf{yy}}} \qquad \boxed{\mathsf{N}_2 = \Sigma_{\mathsf{yy}} - \Sigma_{\mathsf{zz}}}$$

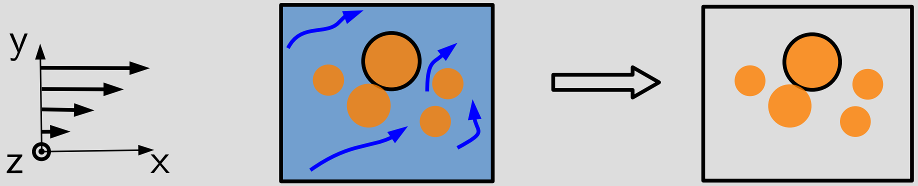
# Numerical methods

Maxey, M. (2017). Simulation methods for particulate flows and concentrated suspensions. Annual Review of Fluid Mechanics, 49(1), 171-193.

- Stokesian dynamics

Brady, J. F., & Bossis, G. (1988). Annual Review of Fluid Mechanics

Sierou, A., & Brady, J. F. (2001). Journal of Fluid Mechanics




No flow is computed

$$\mathcal{F}^h + \mathcal{F}^c = 0 \qquad \begin{pmatrix} \mathcal{F}^h \\ \mathcal{S}^h \end{pmatrix} = -\mathcal{R} \cdot \begin{pmatrix} \mathcal{U} - \mathcal{U}_\infty \\ \mathcal{E}_\infty \end{pmatrix}$$

$$\mathcal{U} = \mathcal{U}_\infty + \mathcal{R}_{\mathcal{FU}}^{-1} \cdot \mathcal{R}_{\mathcal{F}\mathrm{E}} : \mathbf{E}_\infty + \mathcal{R}_{\mathcal{FU}}^{-1} \cdot \mathcal{F}^c \Longrightarrow \text{Particle motion}$$

$$\mathcal{S} = \left[-\mathcal{R}_{\mathcal{SU}} \cdot \mathcal{R}_{\mathcal{FU}}^{-1} \cdot \mathcal{R}_{\mathcal{F}\mathrm{E}} + \mathcal{R}_{\mathcal{S}\mathrm{E}}\right] : \mathbf{E}_\infty - \mathcal{R}_{\mathcal{SU}} \cdot \mathcal{R}_{\mathcal{FU}}^{-1} \cdot \mathbf{F}^c + \frac{1}{2}\left(\Delta\mathbf{r} \otimes \mathbf{F}^c + \mathbf{F}^c \otimes \Delta\mathbf{r}\right) \Longrightarrow \text{Stress}$$

$$\boxed{\mathcal{R} = \mathcal{M}_{\mathrm{ff}}^{-1} + \mathcal{R}_{\mathrm{2B}}^{\mathrm{th}} - \mathcal{M}_{\mathrm{ff,2B}}^{-1}}$$

$\mathcal{M}_{\mathrm{ff}}$ **Far field multipole expansion** (pairwise-additive construction of the mobility tensor)

$\mathcal{R}_{\mathrm{2B}}^{\mathrm{th}}$ **Theoretical 2-body lubrication resistance** (pairwise-additive construction $\Longrightarrow$ sparse matrices)

- Original method : $\mathcal{M}_{\mathrm{ff}}$ is dense $\Longrightarrow$ building $O(N^2)$, inverting $O(N^3)$
- Accelerated SD : O(Nln(N))

**Pros**
- accurate (long range and lub.)
- quite fast   Ouaknin et al (2021). JCP

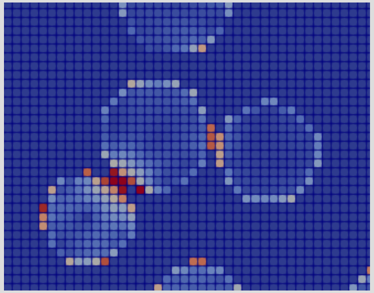
**Cons**
- for spheres only (and maybe spheroids)
- for Newtonian liquid only
- Re = 0 only
- method quite involved

# Fictitious Domain Methods : actual flow computation

- Fluid flow computed over the whole domain
- Particles accounted for using force density λ

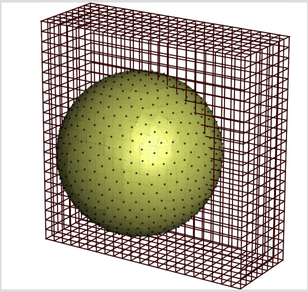
$$\boxed{\begin{aligned} \eta\Delta\mathbf{u} - \nabla \mathrm{p} - \nabla \mathrm{P}_0 + \rho_f \boldsymbol{\lambda} &= 0 \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}}$$

Immersed body



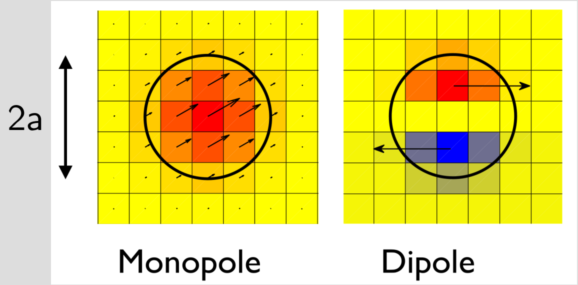
Gallier et al. (2014) JCP.
Orsi et al. (2023) JCP.

Immersed boundary method (IBM)



Uhlmann (2008) Phys. Fluids
Breugem (2012) JCP.

Force coupling method (FCM)

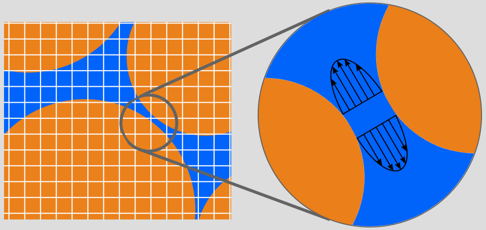



Maxey & Patel (2001) Int. J. Multiphase Flow
Yeo & Maxey (2011) JCP.

- Constraints $\mathbf{x} \in \mathcal{D}_p \quad \mathbf{u}(\mathbf{x}) = \mathbf{U}_p + \boldsymbol{\Omega}_p \times \mathbf{x}$

$$\mathbf{F}^{\mathrm{FDM},p} + \mathbf{F}^{\mathrm{SG},p} + \mathbf{F}^{c,p} = 0$$
$$\mathbf{T}^{\mathrm{FDM},p} + \mathbf{T}^{\mathrm{SG},p} + \mathbf{T}^{c,p} = 0$$

- Subgrid corrections



$$\begin{pmatrix} \mathcal{F}^{\mathrm{SG}} \\ \mathcal{S}^{\mathrm{SG}} \end{pmatrix} = -\mathcal{R}^{\mathrm{SG}} \cdot \begin{pmatrix} \mathcal{U} - \mathcal{U}_\infty(\mathbf{x}_p) \\ -\mathcal{E}_\infty \end{pmatrix}$$

pairwise-additive construction

# FDM : pros and cons

## Pros

- Hydrodynamics : short and long rang interactions included, few restrictive assumptions.
- Sedimentation allowed (long range interactions)
- Extension to any particle shape or liquid
- Extension to Re, St $\neq 0$
- Some open CFD toolboxes (OpenFOAM, Basilisk) with native MPI integration

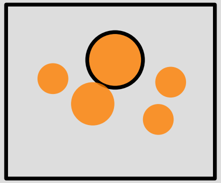
## Cons

- **Not that fast**
- Parallelized particle sub-problem implementation quite difficult

# Distinct (or discrete) elements methods (DEM)

- From granular physics Cundall, P. A., & Strack, O. D. L. (1980). Géotechnique, 29(1), 47-65.
- Molecular dynamics w/o thermal forces, no fluid flow computed

$$M_p \frac{d\mathbf{U}_p}{dt} = \mathbf{F}_p^c + \mathbf{F}^h$$



- Different versions (not a comprehensive list)

  - No inertia, 2-particle lubrication interactions and drag

    Seto et al. (2013) PRL

    Mari et al. (2014) JOR

$$\mathcal{F} = \mathcal{F}_{\mathbf{drag}} - \mathcal{R}_{\mathrm{lub}} \cdot (\mathcal{U} - \mathcal{U}_\infty) + \mathcal{R}'_{\mathrm{lub}} \ \mathbf{:} \ \mathbf{E}_\infty$$

$$\mathbf{F}_p^{\mathrm{drag}} = -6\pi\eta a \left(\mathbf{U}_p - \mathbf{u}_\infty(\mathbf{x}_p)\right)$$

$$\mathbf{T}_p^{\mathrm{drag}} = -8\pi\eta a^3 \left(\mathbf{\Omega}_p - \mathbf{\Omega}_\infty\right)$$

Pairwise additive lubrication

  - Inertia, 2-particle lubrication interactions, drag or no drag (depending on the papers)

    Ness & Sun (2015) PRE

    Cheal & Ness (2016) JOR

$$\mathcal{F} = -\mathcal{R} \cdot \mathcal{U} + (\mathcal{F}_{\mathrm{drag}})$$

  - Inertia, 2-particle lubrication interactions, poromechanical coupling

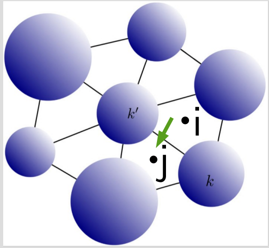
$$\mathcal{F} = -\mathcal{R} \cdot \mathcal{U} + \mathcal{F}_{\mathrm{pore\ P.}}$$

Chareyre et al. (2012] Transp. in porous media

Marzougui et al. (2015) Granular Matter

Chevremont (2019) Phys.rev. Fluids

Adapted from Marzougui et al. (2015) Granular Matter

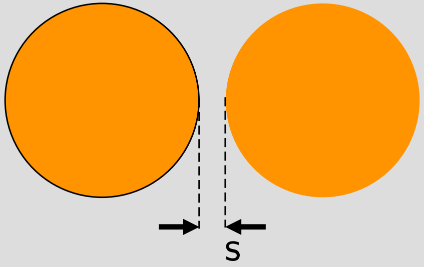



$$q_{ij} \propto P_i - P_j$$

- Pressure velocity coupling $\frac{dV_i}{dt} = \sum_j q_{ji}$
- Viscous force $\mathcal{F}_{\mathrm{Pore\ P.}} = \mathbf{I}^f \cdot \mathcal{P}$

# DEM

- All versions : cutoff distance for lubrication interactions $s/a \gtrsim 0.1 \Rightarrow \mathcal{F} = 0$




Pros

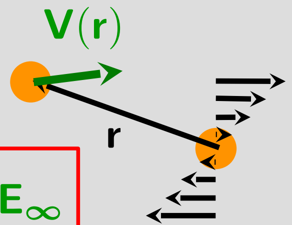
- **Fast** (meaning large size systems, large volume fractions ...)
- Some open codes, **easy to use** (LAMMPS (Ness), YADE (Chareyre et al.)...)

Cons

- Only lubrication interactions :
  - ➔ Better for high volume fraction
  - ➔ Not suitable for sedimentation
  - ➔ Specific strategies for some problems involving heterogeneous flow, e.g. migration (Kolmogorov flow = external force on the particles), resuspension...
  - ➔ Data to be checked against method with more comprehensive hydrodynamics (migration, velocity fluctuations etc.)

# A force free particle pair in simple shear flow

- Resistance matrices known exactly (semi-analytical)
- Direct integration of the momentum balance equation

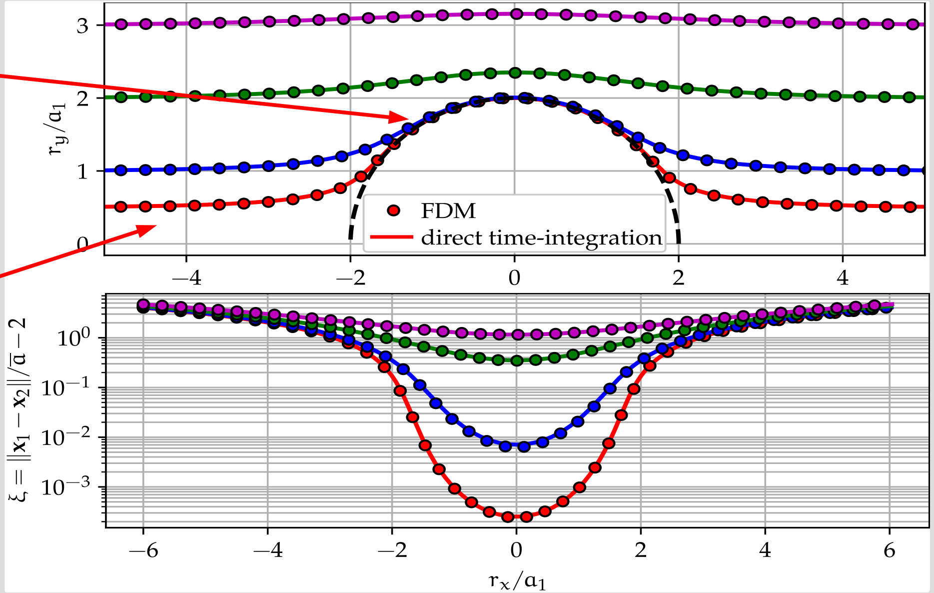



$$0 = -\mathcal{R}_{\mathcal{F}\mathcal{U}} \cdot (\mathcal{U} - \mathcal{U}_\infty) + \mathcal{R}_{\mathcal{F}\mathrm{E}} : \mathbf{E}_\infty$$

$$\boxed{(\mathbf{U}_1,\, \mathbf{U}_2,\, \boldsymbol{\Omega}_1,\, \boldsymbol{\Omega}_2) = \mathcal{U}_\infty + \mathcal{R}^{-1}_{\mathcal{F}\mathcal{U}}(\mathbf{r}) \cdot \mathcal{R}_{\mathcal{F}\mathrm{E}}(\mathbf{r}) : \mathbf{E}_\infty}$$

Short range Interactions (lubrication)

Long range interactions




*Orsi, M. (2022) Doctoral dissertation, Université Côte d'Azur.*

Lubrication

No contact
but

$$\frac{r_{min}}{a} \approx 4.10^{-5}$$

Arp, P. A. & Mason, S. G. (1977).
J. Colloid Interface Sc.

$$a = 40\mu m \Rightarrow r_{min} \approx 1.6.10^{-9} m$$

Asperities = contact !

# No direct interaction force in NB suspensions: lubrication breakdown

Melrose, J. R., & Ball, R. C. (1995). The pathological behaviour of sheared hard spheres with hydrodynamic interactions. Europhysics letters, 32(6), 535.

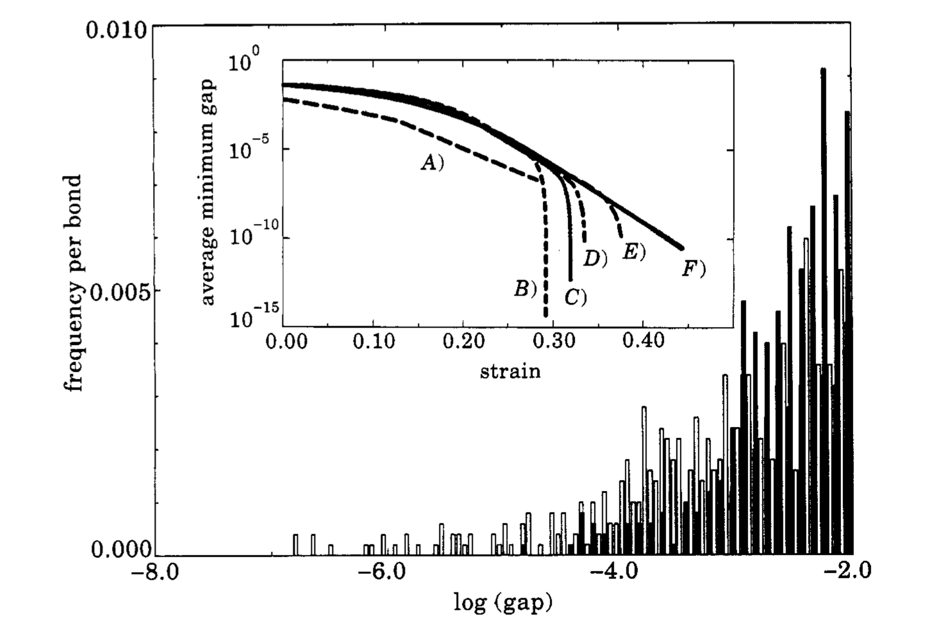
Ball, R. C., & Melrose, J. R. (1995). Lubrication breakdown in hydrodynamic simulations of concentrated colloids. Advances in colloid and interface science, 59, 19-30.

- Simplified Stokesian dynamics (DEM, no far field)
- Various time schemes (4th order RK, 2nd order predictor corrector ) and time steps
- Unphysical value of the gap

$$d = 40\mu m \;\Rightarrow\; s = 4.10^{-15} m$$

- No steady state (viscosity)
- Simulation stops (too long, gap collapse)
- Also observed in full SD

Dratler, D. I., & Schowalter, W. R. (1996). Journal of Fluid Mechanics




Melrose, J. R., & Ball, R. C. (1995). Europhysics letters, 32(6), 535.

RK. $N_p$=700 (A), $N_p$=50 (F)
PC. $N_p$= 50. dt = $5.10^{-4}$ (B), dt = $5.10^{-4}$ (C), dt = $5.10^{-5}$ (D), dt = $1.10^{-5}$ (E)

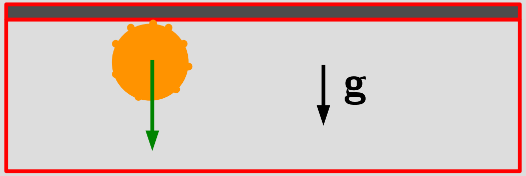
Unphysical
Computationally Intractable

# Contact due to roughness in NB suspensions : quite an old idea (1980-1990)

- Breaking of periodic orbit of a particle pair in shear flow — Arp, P. A. & Mason, S. G. (1977) J. Colloid Interface Sci.

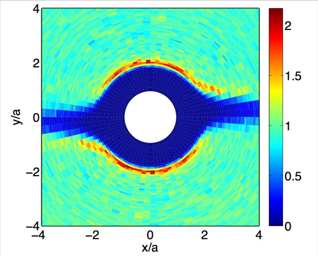
- Proposed to explain particle migration — Leighton, D., & Acrivos, A. (1987) Journal of Fluid Mechanics

- Measured via particle-wall hydro interactions — Smart & Leighton (1989).Physics of Fluids A: Fluid Dynamics




Time for separation = $f(h_r)$ ⟺ Lubrication cutoff

- Roughness connected to asymmetric pair distribution function — Rampall et al. (1997). Journal of Fluid Mechanics




Blanc, F. et al. (2011). Physical review letters, 107(20), 208302.

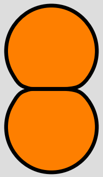
# Contact force modelling

- Contact between solid bodies : complicated matter

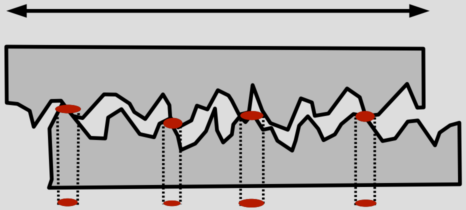
Johnson, K. L. (1987). Contact mechanics. Cambridge university press.

- Suspensions: many parameters (Jean Comtet lecture)
  - Smooth / rough surface
  - Elastic / plastic / viscoelastic solids
  - Adhesion
  - Surface physico-chemistry (grafted polymers, electric charge)

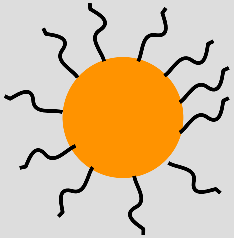
smooth

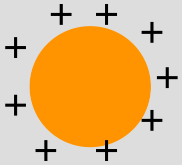


rough



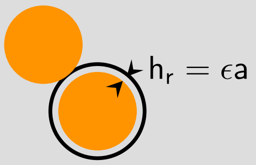
$A_{real} \ll A_{apparent}$

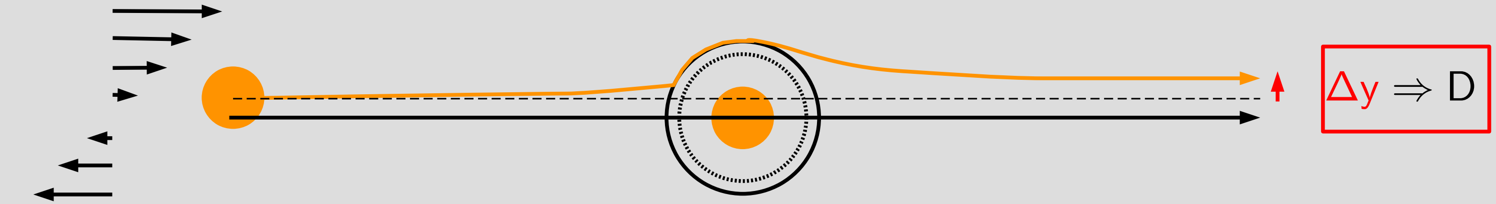




# Contact forces in simulations (NB suspensions) : the early days

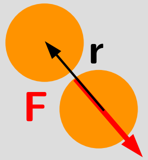
- Roughness (contact) induces irreversibility — Da Cunha, F. R., & Hinch, E. J. (1996). J.F.M
  - Shear-induced diffusivity in dilute suspensions (2 particles)
  - Simple model of roughness : modification of the mobility functions equivalent to a radial force




$$s = r - 2a = h_r \Rightarrow V_r = 0$$

- Repulsive spherical potential
  - Avoid lubrication breakdown
  - Weak influence of the specific form if short range (w.r.t radius a)




**Double layer-like**

$h_r \sim 1/\tau$

$$\mathbf{F} = 2f_0\tau \frac{e^{-\tau s}}{1 + e^{-\tau s}} \frac{\mathbf{r}}{\|\mathbf{r}\|}$$

Dratler, D. I., & Schowalter, W. R. (1996). J.F.M, 325, 53-77.

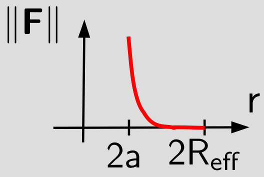
$$\mathbf{F} = f_0\tau \frac{e^{-\tau s}}{1 - e^{-\tau s}} \frac{\mathbf{r}}{\|\mathbf{r}\|}$$

Sierou, A., & Brady, J. F. (2002). J. Rheol. 46(5), 1031-1056.

**Arbitrary**

$$r < R_{\text{eff}} \Rightarrow \mathbf{F} = -6\pi\eta a V_{\text{ref}} \left( \frac{R_{\text{ref}}^2 - \|\mathbf{r}\|^2}{R_{\text{ref}}^2 - a^2} \right)^6 \frac{\mathbf{r}}{\|\mathbf{r}\|}$$

Yeo, K., & Maxey, M. R. (2010).Journal of Fluid Mechanics, 649, 205-231.




Cut off distance

$R_{\text{eff}}/a = 2.002$

$h_r = R_{\text{eff}} - 2a = 0.002a$

# Further modelling : elasto-frictional contact

- Standard in dry granular physics

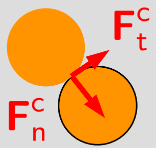
Cundall, P. A., & Strack, O. D. L. (1980). Géotechnique

Shäfer, J., Dippel, S., & Wolf, D. E. (1996). Journal de physique I

- More recently in suspensions

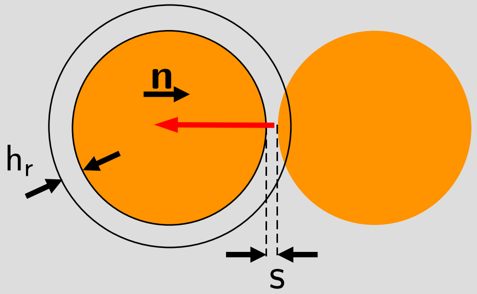
Seto, R., Mari, R., Morris, J. F., & Denn, M. M. (2013) PRL

Gallier, S., Lemaire, E., Peters, F., & Lobry, L. (2014). JFM




Point forces

- Elastic normal force




Compression $\delta = h_r - s$

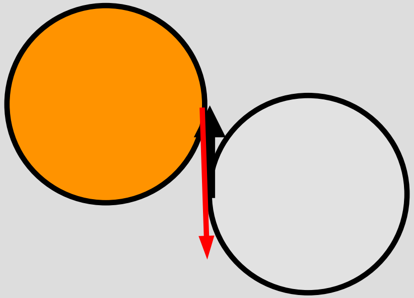
$s < h_r$
$\delta > 0$ $\Rightarrow$ $\mathbf{F}^c_n = -k_n \delta \, \mathbf{n}$ Linear spring

(Hertz $k_n \propto \sqrt{\delta}$ )

No need for dissipative terms $-\gamma_n \dot{\delta}$ $-\gamma_t \dot{\mathcal{Y}}$

if lubrication is kept

- Elastic tangential force



Slip velocity $\mathbf{U}_s = \mathbf{U}_i - \mathbf{U}_j - [(\mathbf{U}_i - \mathbf{U}_j) \cdot \mathbf{n}] \cdot \mathbf{n} + (a_i \boldsymbol{\Omega}_i + a_j \boldsymbol{\Omega}_j) \times \mathbf{n}$

Accumulated slip $\mathcal{Y} = \int_0^t \mathbf{U}_s \, \mathrm{d}t$

Elastic force (stick phase) $\mathbf{F}^c_t = -k_t \mathcal{Y}$ (Hertz-Mindlin $k_t \propto \sqrt{\delta}$ )

Slip condition (Amontons-Coulomb) $||\mathbf{F}^c_t|| \geq \mu_s ||\mathbf{F}^c_n||$ $\Rightarrow \mathbf{F}^c_t = \mu_s \, ||\mathbf{F}^c_n|| \dfrac{\mathbf{F}^c_t}{||\mathbf{F}^c_t||}$

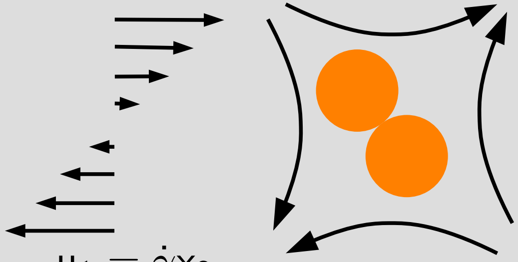
# Dimensionless numbers

- Equations made dimensionless

Length : radius a

Time : $\dot{\gamma}^{-1}$

Force : $6\pi\eta a^2\dot{\gamma}$



$u_1 = \dot{\gamma} x_2$

Two particles
= dilute suspension

$F \sim 6\pi\eta a^2\dot{\gamma} = 6\pi\Sigma_{12}a^2$

Nir, A., & Acrivos, A. (1973). JFM

Volume fraction $\phi = \dfrac{V_s}{V_{tot}}$

Roughness $h_r/a \sim 10^{-3} - 10^{-2}$ Experiments with model spheres

Shear rate

$$\dot{\Gamma}_L = \frac{6\pi\eta a^2\dot{\gamma}}{k_n h_r}$$

$$\dot{\Gamma}_H = \frac{6\pi\eta a^2\dot{\gamma}}{k_n h_r^{3/2}}$$

$$F_n^c(\bar{\delta}) \sim 6\pi a^2\Sigma_{12} \Longleftrightarrow \boxed{\begin{array}{l} \left(\dfrac{\bar{\delta}}{h_r}\right)_L \sim \dot{\Gamma}_L \\ \left(\dfrac{\bar{\delta}}{h_r}\right)_H \sim \dot{\Gamma}_H^{2/3} \end{array}}$$

Increases with
volume fraction or stress

N.B. Shear stress : $\eta_r\dot{\Gamma}$

Typical compression (dilute case)

Tangential stiffness $k_t/k_n$

Friction coefficient $\mu_s$

# Stiffnesses

- A simple estimation: a sphere (a) against an asperities ($h_r$ << a) $F_n^c = \frac{2}{3}\frac{E}{1-\nu^2} h_r^{1/2}\delta^{3/2}$

  Peters et al. (2016). Journal of rheology,

PMMA

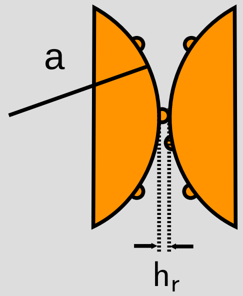
$E \sim 3\mathrm{GPa}$ $\qquad a \sim 20\mu m$

$\nu \sim 0.4$ $\qquad h_r/a \sim 10^{-2}$

$\eta \sim 1\mathrm{Pa.s}$

$\dot\gamma \sim 100 s^{-1}$

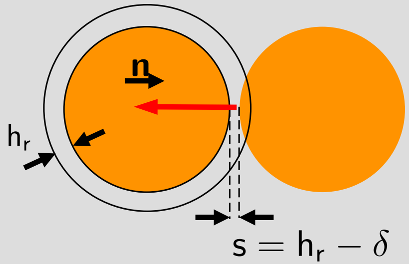
$$\boxed{\begin{array}{c}\dot\Gamma \sim 8.10^{-3} \\ \bar\delta/h_r \sim 4.10^{-2}\end{array}}$$




- $\dot\Gamma \leq 10^{-1}$ Mostly Newtonian

  $\dot\Gamma > 10^{-1}$ Weak shear thickening behaviour

  Gallier et al. (2014). Journal of Fluid Mechanics




$\Sigma_{12} \nearrow \Rightarrow \delta \nearrow$

$\Rightarrow$ lubrication $\nearrow$

- One strategy : keeping $\begin{array}{l}\langle\delta/a\rangle \leq \delta_{max}/a \sim 5\% \\ \langle\mathcal{Y}/a\rangle \leq \mathcal{Y}_{max}/a \sim 5\%\end{array}$ and $\begin{array}{c}k_n \\ k_t\end{array} \propto \dot\gamma$ $\qquad \eta_s\left(\phi, h_r, \cancel{6\pi a^2\eta\dot\gamma/F_n^c(h_r)}\right)$

  Mari et al. (2014). J. Rheol.

No stress scale other than $\eta\dot\gamma$ $\Longrightarrow$ $\boxed{\approx \text{ Newtonian behavior}}$ $\qquad \dot\Gamma$

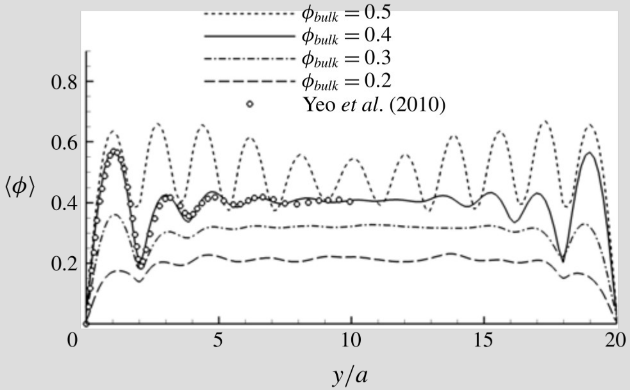
$\dot\Gamma = \mathrm{cst} \Longleftrightarrow \bar\delta \sim \mathrm{cst}$

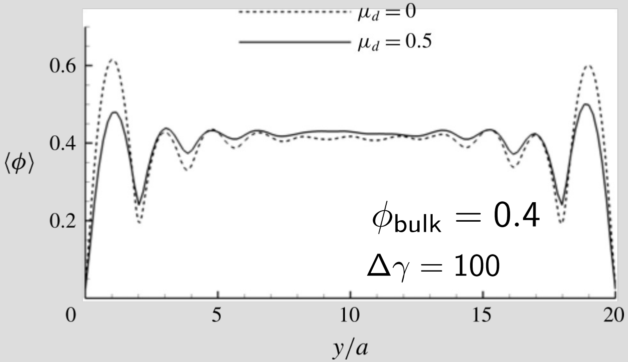
- Usually $k_t/k_n = 2/7$

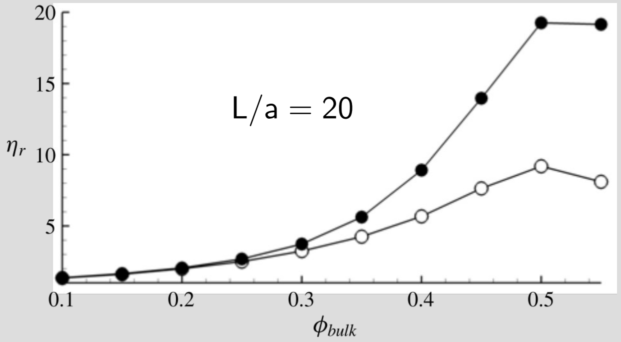
# Wall effects

## Monodisperse supensions

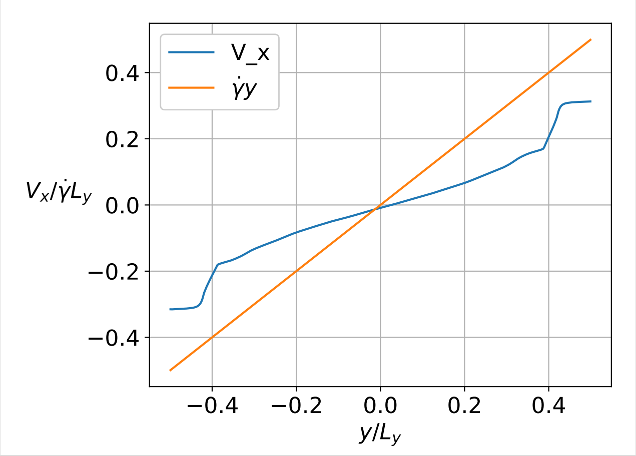
Gallier et al. (2016). J. Fluid Mech.

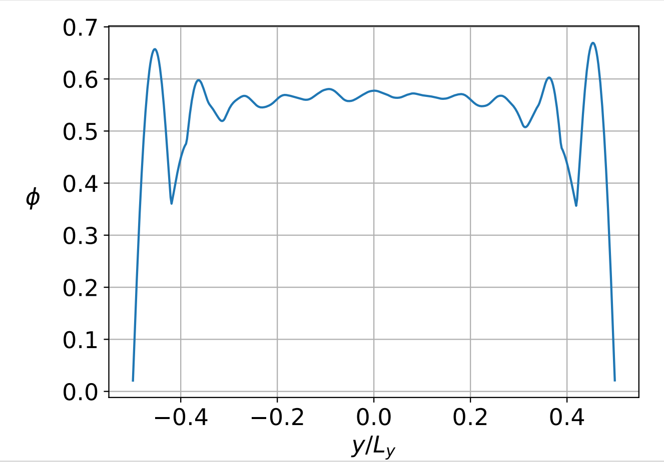



## Slight bidispersity




$\mu_s = 0.5$

$\overline{\phi} = 0.54$

$a_2/a_1 = 1.4$

$\overline{\phi_1} = \overline{\phi_2} = \overline{\phi}/2$

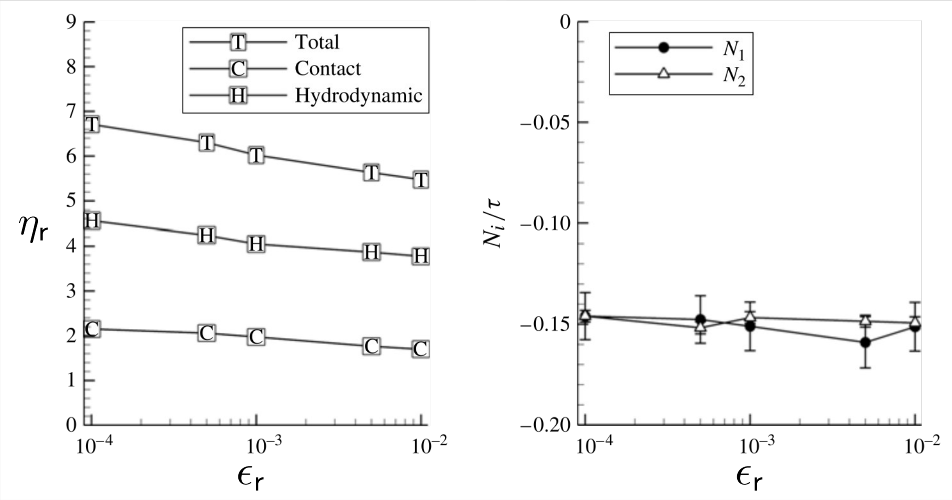
Same Data as in
Michel Orsi (2023) JCP.

# Roughness

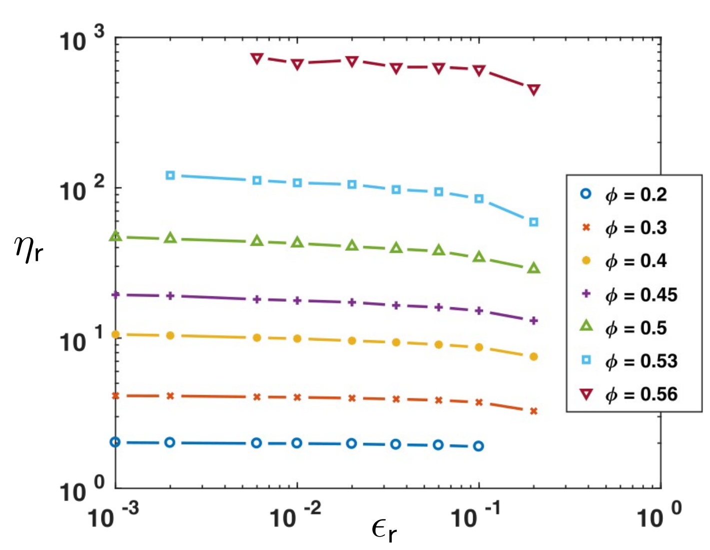
$\epsilon_r = h_r/a$

$\mu_s = 0$

$\phi = 0.4$




Gallier et al. (2014). J. Fluid Mech.

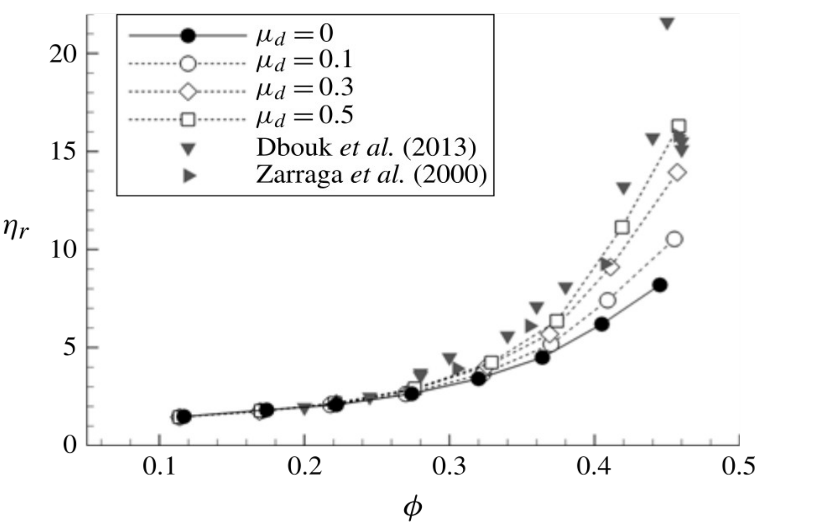



- $h_r$ as such : weak influence
- $h_r \searrow \;\Rightarrow$ lubrication $\nearrow$

  $\Rightarrow \eta_r \nearrow$
- $h_r$ acts as a lubrication cut-off

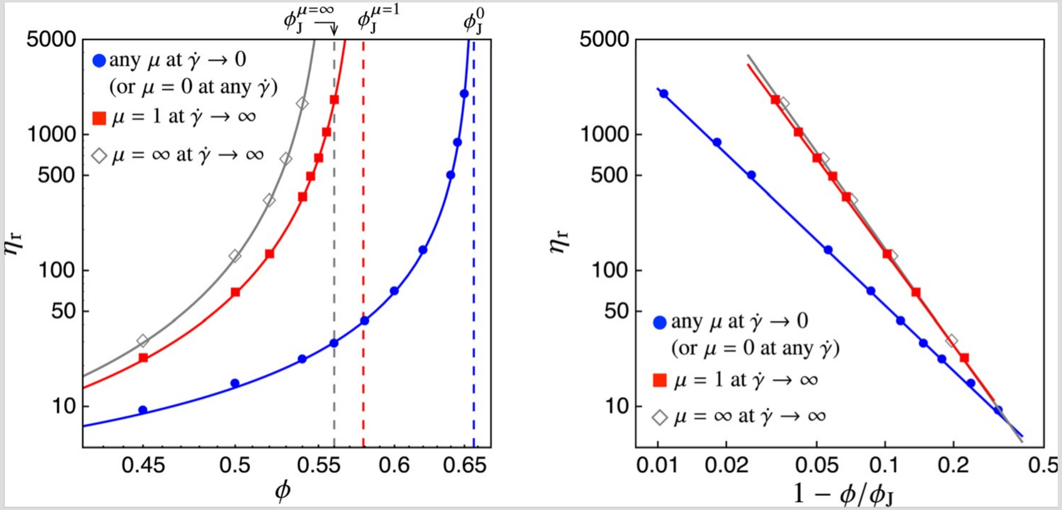
Chevremont et al. (2019). Phys. Rev. Fluids

# Friction coefficient




Gallier et al. (2014). J. Fluid Mech.

$$\boxed{\mu_S = 0,5 \Rightarrow \eta_r^{\text{simulation}} \approx \eta_r^{\text{exp}}}$$




Mari et al. (2014). J. Rheol.

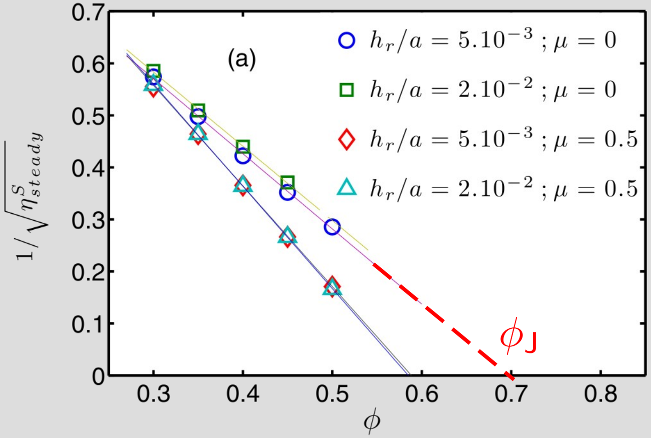
$$\boxed{\mu_S \nearrow \Rightarrow \phi_J \searrow \Rightarrow \eta_r \nearrow}$$

Close to jamming

$$\eta_r(\mu_s = 0) \propto (1 - \phi/\phi_J)^{-1.6} \quad \phi_J \approx 0.66$$

$$\eta_r(\mu_s = 1) \propto (1 - \phi/\phi_J)^{-2.3} \quad \phi_J \approx 0.58$$

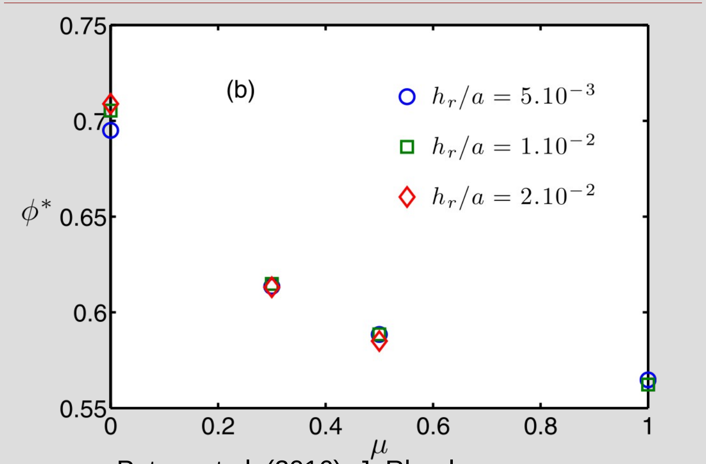
# Correlation laws



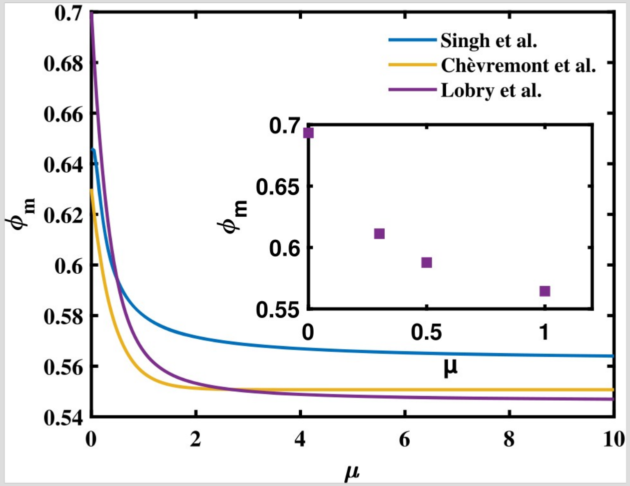

$$\boxed{\eta_r = \frac{\alpha}{(1-\phi/\phi_J)^2}}$$

Maron-Pierce

- $\mu_S \nearrow \Rightarrow \phi_J \searrow \Rightarrow \eta_r \nearrow$
- $\phi_J(\mu_s = 0) \approx 0.7$ too large




Peters et al. (2016). J. Rheol.

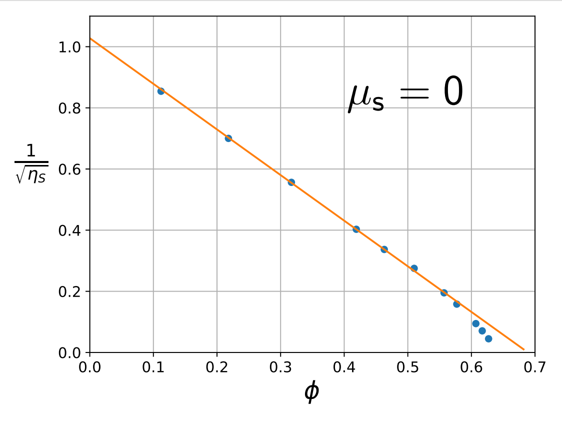



Lemaire et al. (2023) Rheologica Acta

Singh et al. (2018). JOR

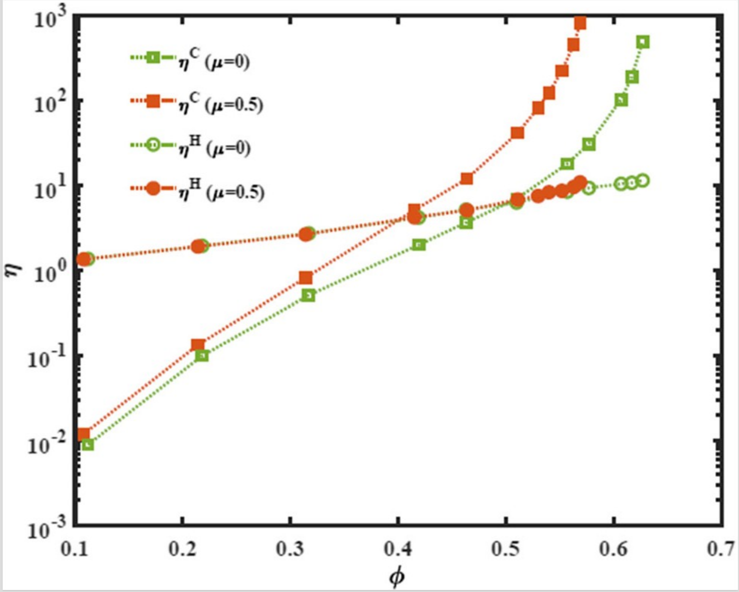
Chevremont et al. (2019). Phys. Rev. Fluids

Lobry et al. (2019). JFM




data from Gallier al. (2018). Phys. Rev. Fluids

# Contact vs. hydrodynamics

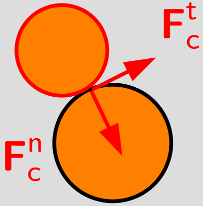


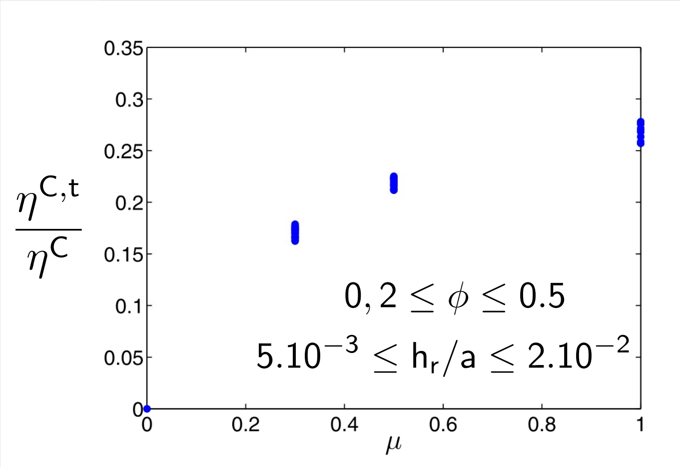

Lemaire et al. (2023). Rheologica Acta.

data from Gallier al. (2018). Phys. Rev. Fluids

- $\eta^{H} = f(\phi, \cancel{\mu_s})$
- $\mu_S \nearrow \Rightarrow \eta^C \nearrow$
- $\mu_s = 0 \quad \eta^c = \eta^h$ at $\phi \approx 0.5$
  $\mu_s = 0.5 \quad \eta^c = \eta^h$ at $\phi \approx 0.4$

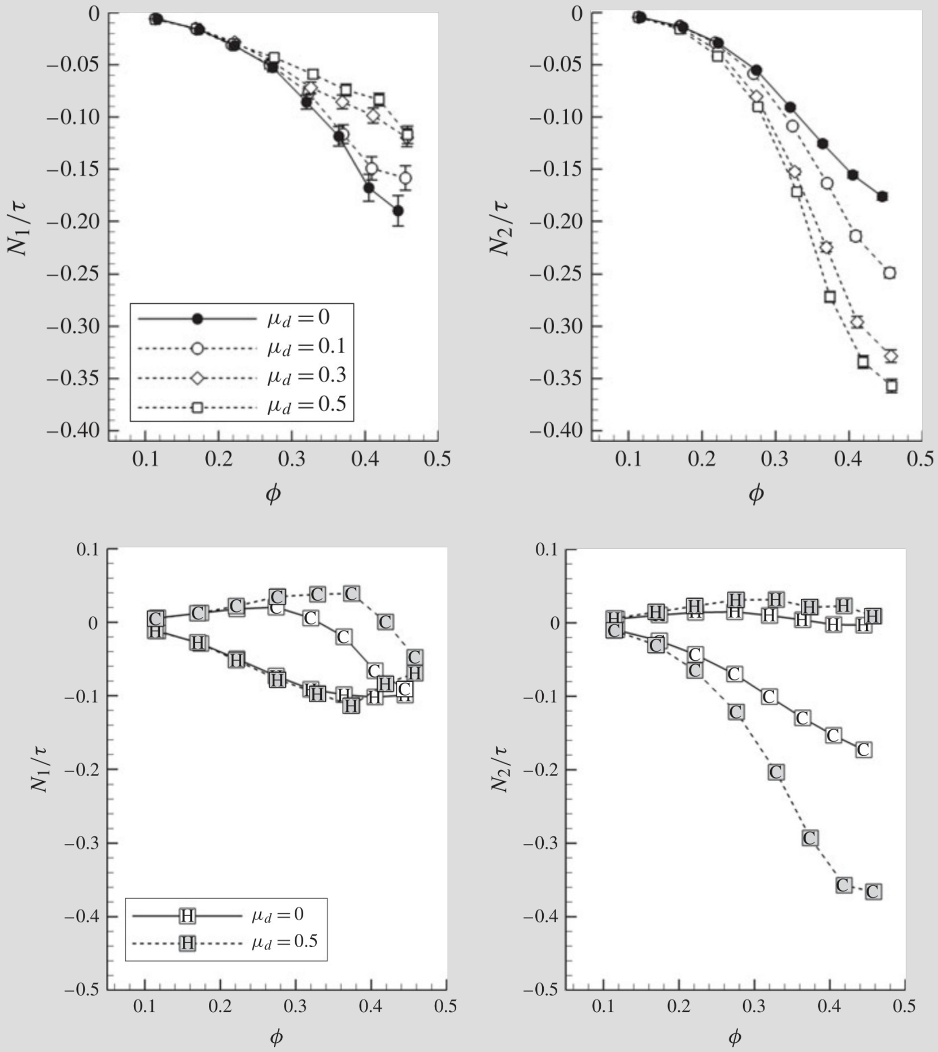
- $\mu_S \nearrow \Rightarrow \begin{matrix} \eta^{C,n} \nearrow \\ \eta^{C,t} \nearrow \end{matrix}$
- $\dfrac{\eta^{C,t}}{\eta^C} = f(\mu_s)$




Peters et al. (2016). J. Rheol.

# Normal stresses

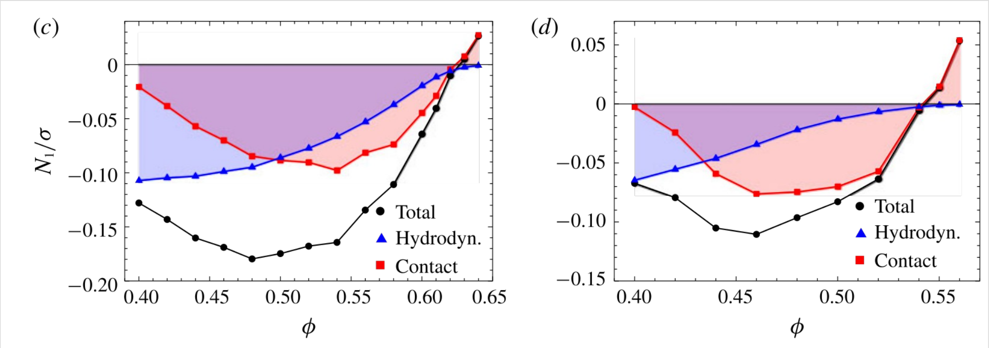



Gallier et al. (2014). J. Fluid Mech.

- $\mu_S = 0 \Rightarrow \frac{N_1}{\tau} \approx \frac{N_2}{\tau} \leq 0$
- $\mu_S \nearrow \Rightarrow \begin{cases} \left|\frac{N_1}{\tau}\right| \searrow \\ \left|\frac{N_2}{\tau}\right| \nearrow \end{cases}$

Seto & Giusteri (2018). J. Fluid Mech.

$\mu = 0$ $\mu = 1$

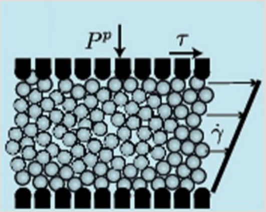



N1 >0 : finite particle stiffness

# Pressure imposed rheology

Born à Marseille (like Ricard, only tastier)

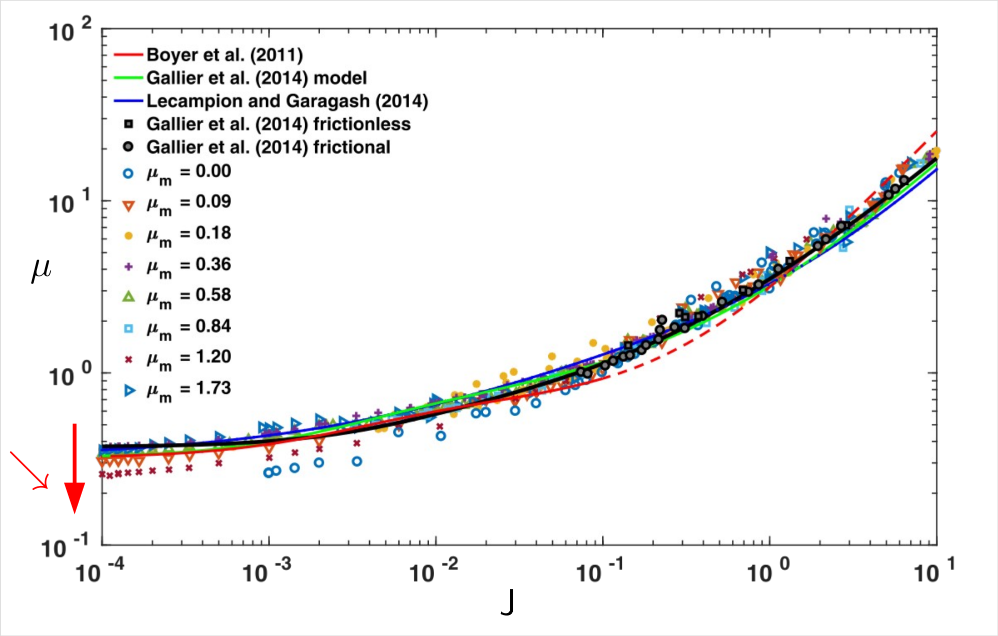
Boyer, F., Guazzelli, É., & Pouliquen, O. (2011) Unifying suspension and granular rheology. PRL.




$$\mathsf{J} = \frac{\eta\dot{\gamma}}{\Sigma^{\mathsf{c}}_{22}} \qquad \mu = \frac{\Sigma_{12}}{\Sigma^{\mathsf{c}}_{22}} = \mu(\mathsf{J}, \mu_{\mathsf{s}})$$

$$\phi/\phi_{\mathsf{J}} = \mathsf{f}(\mathsf{J}, \mu_{\mathsf{s}}) \approx \mathsf{f}(\mathsf{J})$$

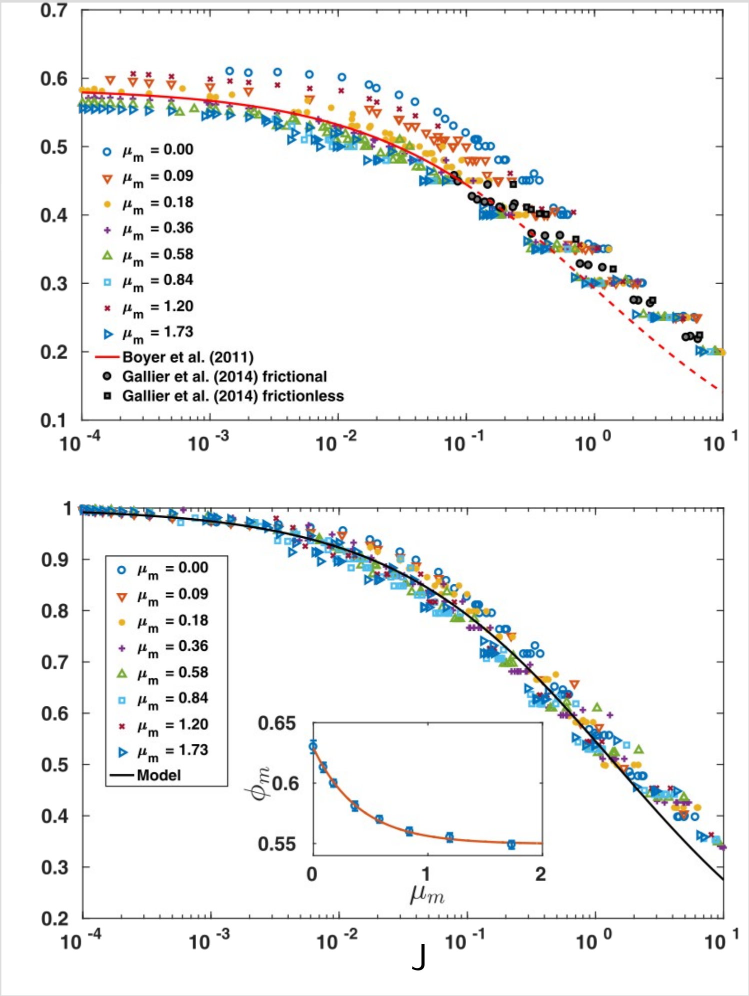
$\mu_{\mathsf{m}}$




Chevremont et al. (2019). Phys. Rev. Fluids

$\phi$

$\phi/\phi_{\mathsf{J}}$




$\mu_{\mathsf{c}} = \lim_{\mathsf{J}\to 0} \mu \sim 0.3 - 0.4$ Independent of $\mu_m$ if $\mu_{\mathsf{m}} \gtrsim 0.3 - 0.4$

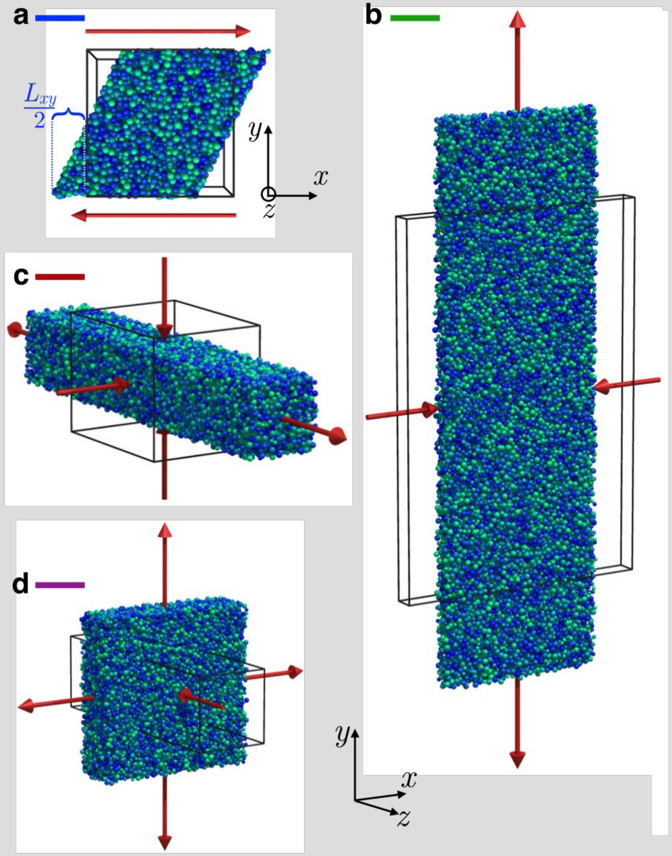
$\mu_{\mathsf{c}}(\mu_{\mathsf{m}} = 0) \sim 0.1$

# Extensional rheology

Cheal, O., & Ness, C. (2018). J. Rheol.

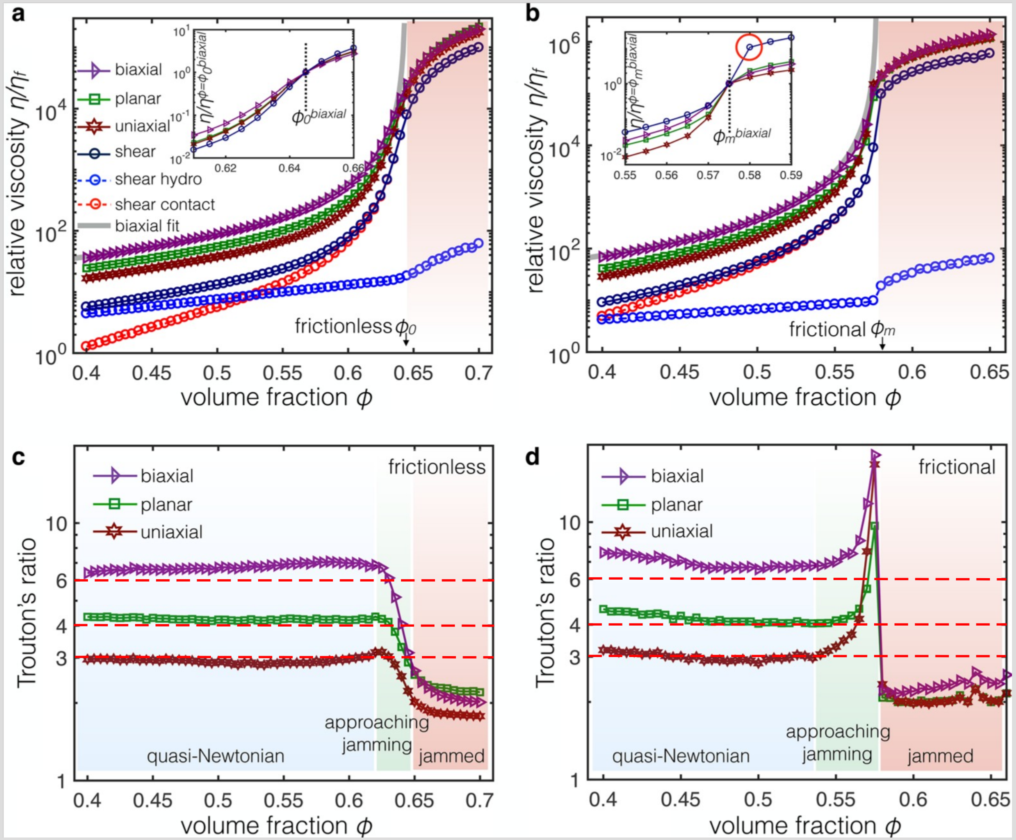
shear

uniaxial




planar

biaxial




$$\boxed{\eta_E = \frac{\Sigma_{ext} - \Sigma_{dilat}}{\dot{\epsilon}_{ext}}}$$

Seto

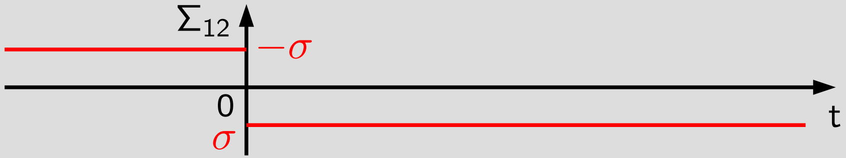
Trouton's ratio

$$\boxed{Tr = \frac{\eta_E}{\eta_S}}$$

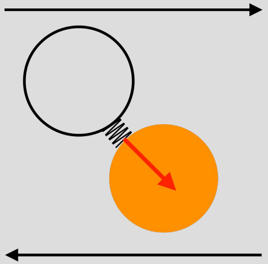
Newtonian - - -

# Shear reversal

Standard splitting H/C

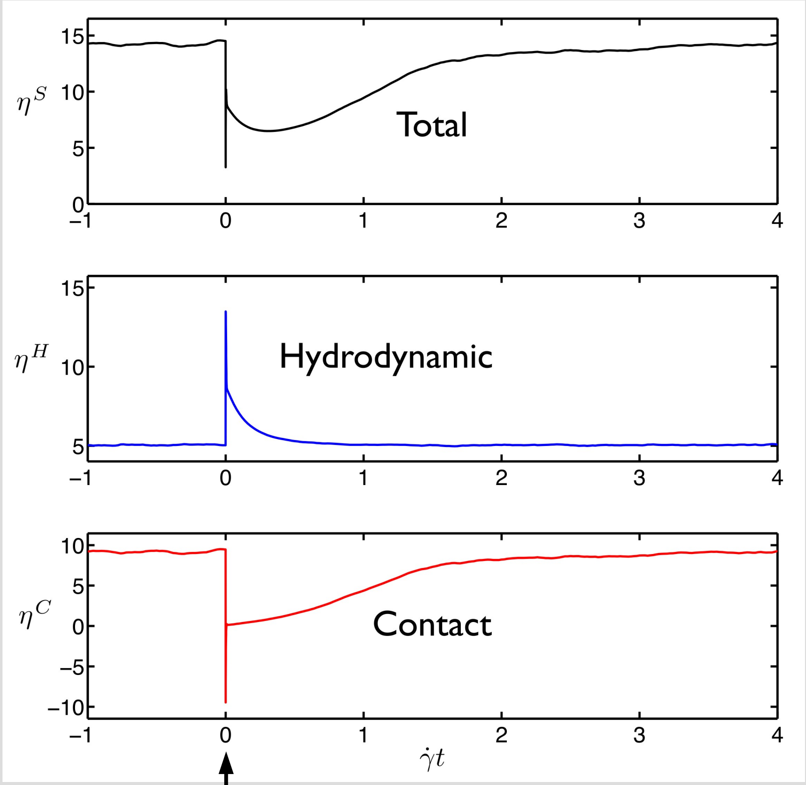



$\dot{\gamma}t = 0^-$



Positive contact contribution

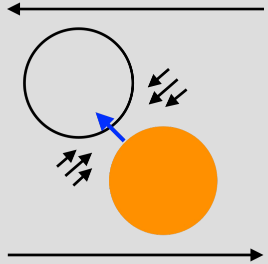
Low lubrication contribution




Steady shear | « Slow » transient

Fast force relaxation

$\dot{\gamma}t \sim \dot{\gamma}\tau_{relax}$



Contact contribution ≈ 0

High and positive lubrication contribution

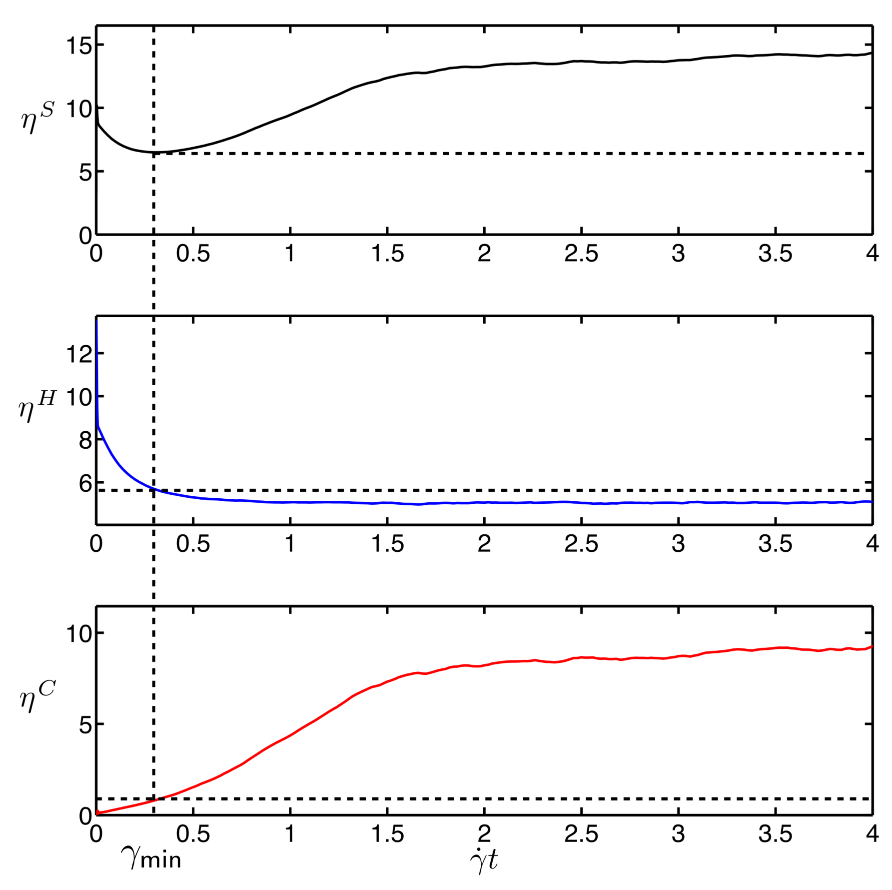
$\phi = 0.45$

$\mathrm{h_r}/\mathrm{a} = 5.10^{-3}$

$\mu_s = 0.5$

Peters et al. (2016) JOR

# Shear reversal : measuring $\eta^{C}$



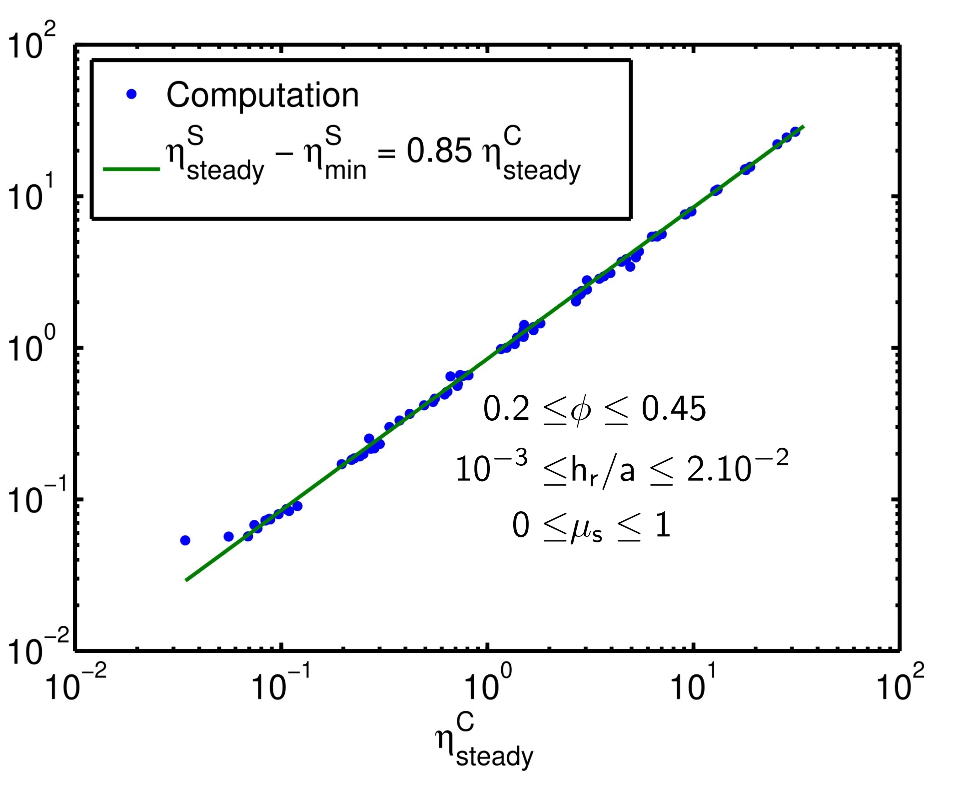

$\eta^{S}_{\text{steady}} - \eta^{S}_{\text{min}}$

$\gamma = \gamma_{\text{min}}$ $\quad \eta^{H} \approx \eta^{H}_{\text{steady}}$, $\eta^{C} \approx 0$ $\Rightarrow$ $\boxed{\eta^{S}_{\text{min}} \approx \eta^{H}_{\text{steady}}}$

$$\boxed{\eta^{S}_{\text{steady}} - \eta^{S}_{\text{min}} \approx \eta^{S}_{\text{steady}} - \eta^{H}_{\text{steady}} = \eta^{C}_{\text{steady}}}$$

# Shear reversal : measuring $\eta^C$

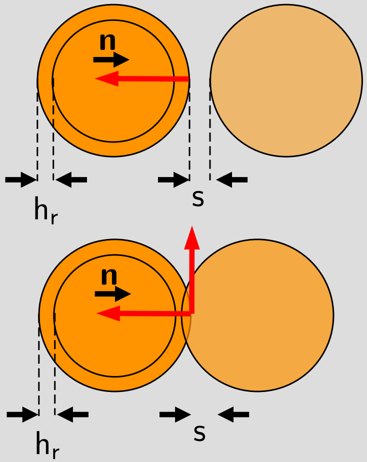
$$\eta^S_{steady} - \eta^S_{min} = f(\eta^C_{steady})$$




Peters et al. (2016) JOR

Seto et al. (2013) PRL 111(21), 218301.
Mari et al. (2014). J. Rheol., 58(6), 1693-1724.

- Direct interaction forces : short range repulsive + contact

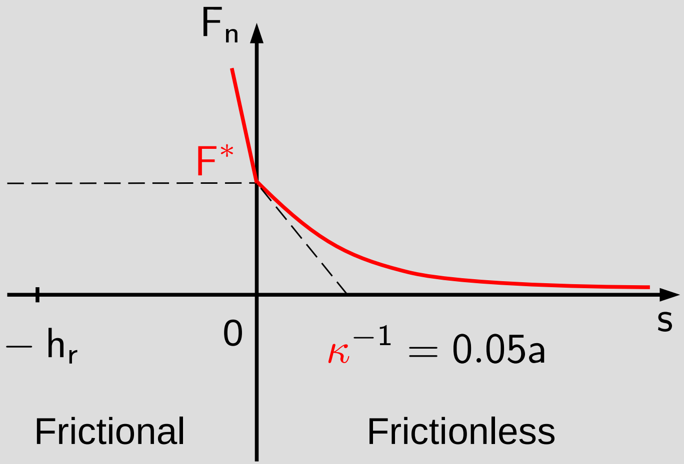



Double layer-like    frictionless

$$s > 0 \;\Rightarrow\; \mathbf{F}_n = -F^* e^{-\kappa s}\,\mathbf{n}$$
$$\mathbf{F}_t = 0$$

Contact    frictional

$$s < 0 \;\Rightarrow\; \mathbf{F}_n = (k_n s - F^*)\;\mathbf{n}$$
$$\mathbf{F}_t = -k_t \mathcal{Y}$$

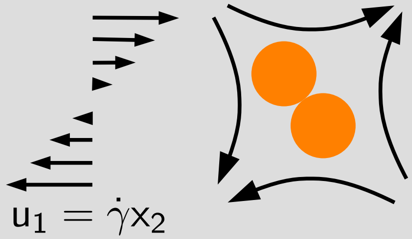



→ Sliding criteria $\boxed{\dfrac{\|\mathbf{F}_t\|}{\|\mathbf{F}_n\| - F^*} \geq \mu_s}$

- Stress scale $\sigma_0 = \dfrac{F^*}{6\pi a^2}$    $\dot{\gamma}_0 = \dfrac{\sigma_0}{\eta_0}$

Contact if $6\pi a^2 \Sigma_{12} \gtrsim F*$

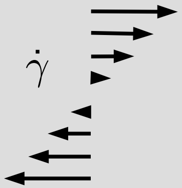
$\boxed{\Sigma_{12} \gtrsim \sigma_0}$

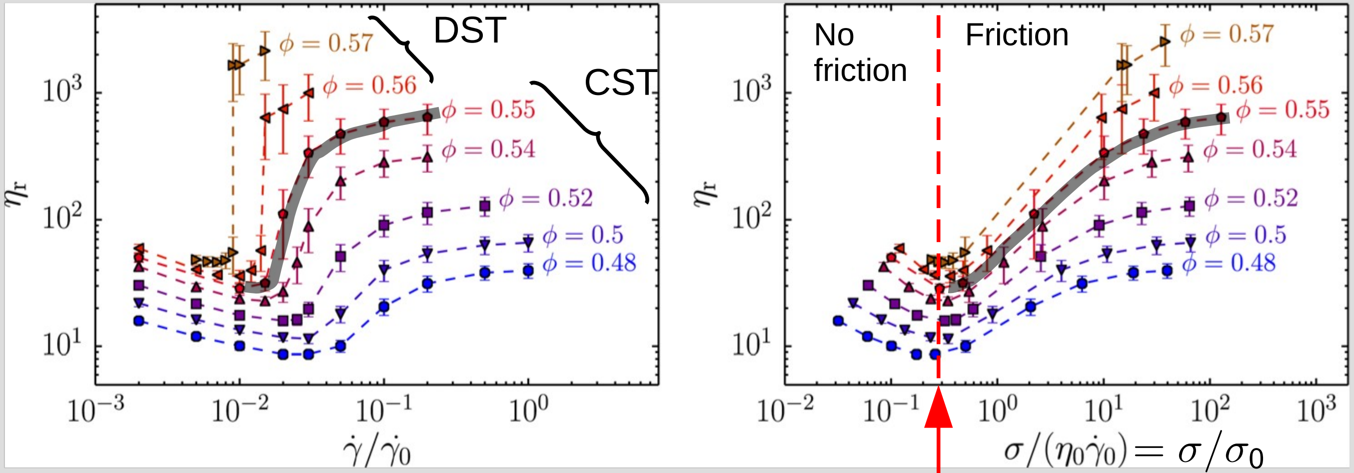



- Roughness = lubrication cut-off    $h_r/a = 10^{-3}$
- DEM (high ϕ), bidisperse $a_2/a_1 = 1.4$

# Continuous / discontinuous shear thickening : lubricated to frictional transition

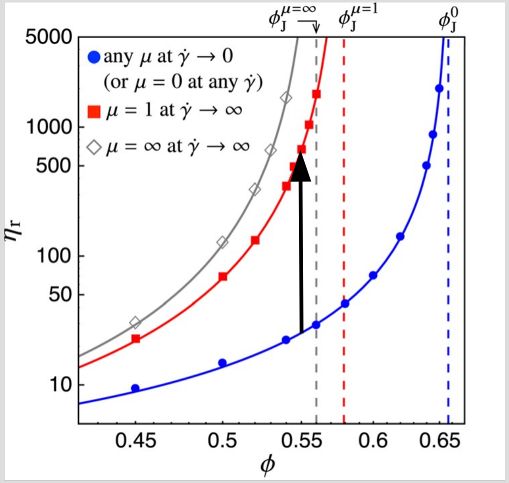
Controlled shear rate



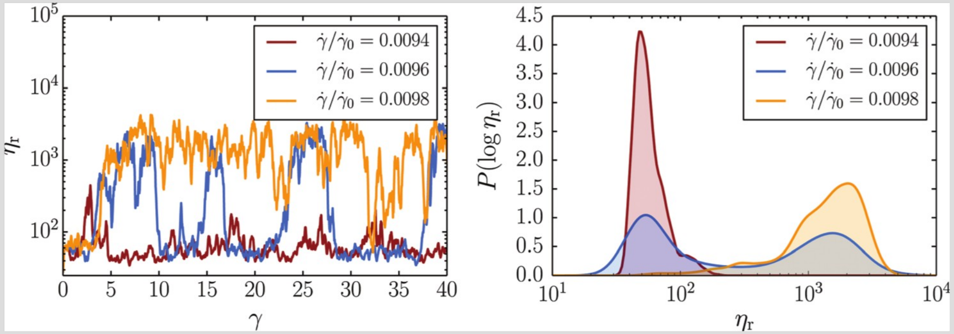




Mari et al. (2014). J. Rheol.

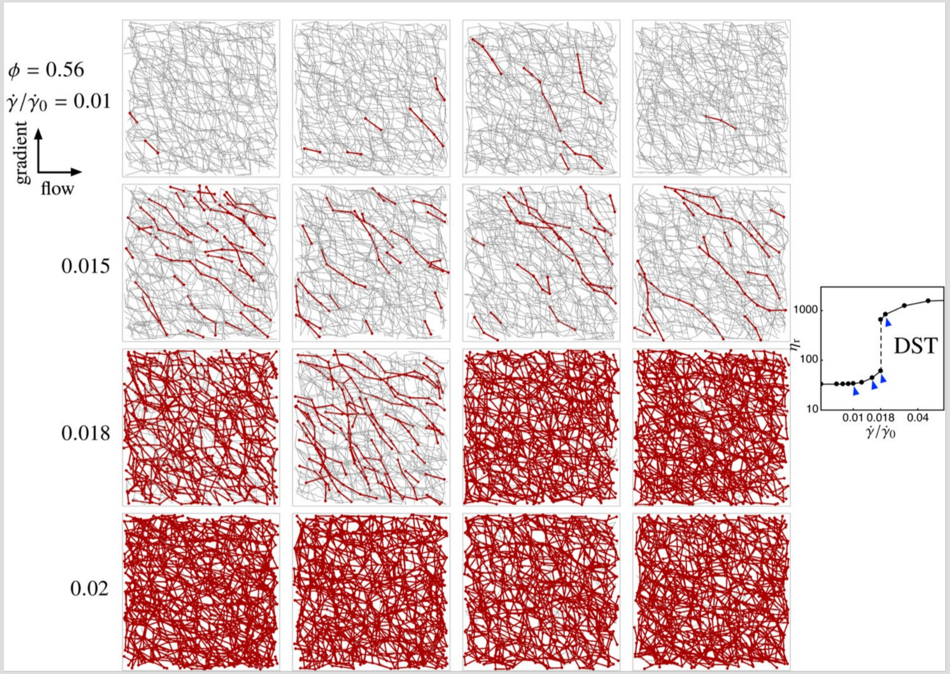
ST inception $\sigma \sim 0.3\sigma_0$

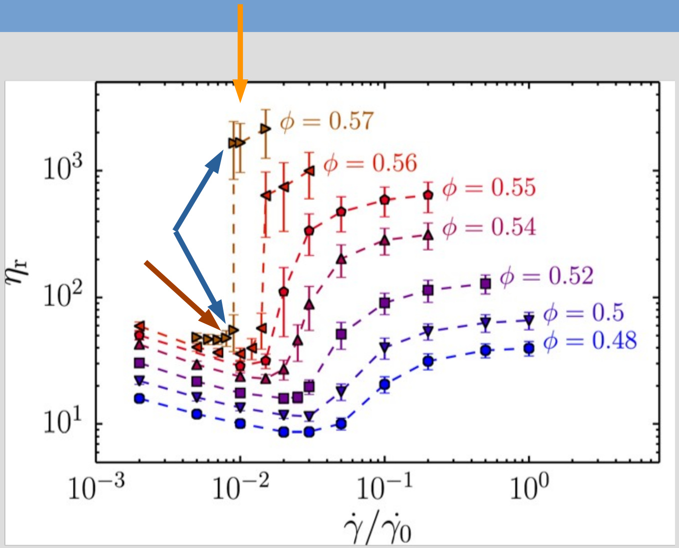


# Intermittency in DST transition (controlled rate)




Mari et al. (2014). J. Rheol.

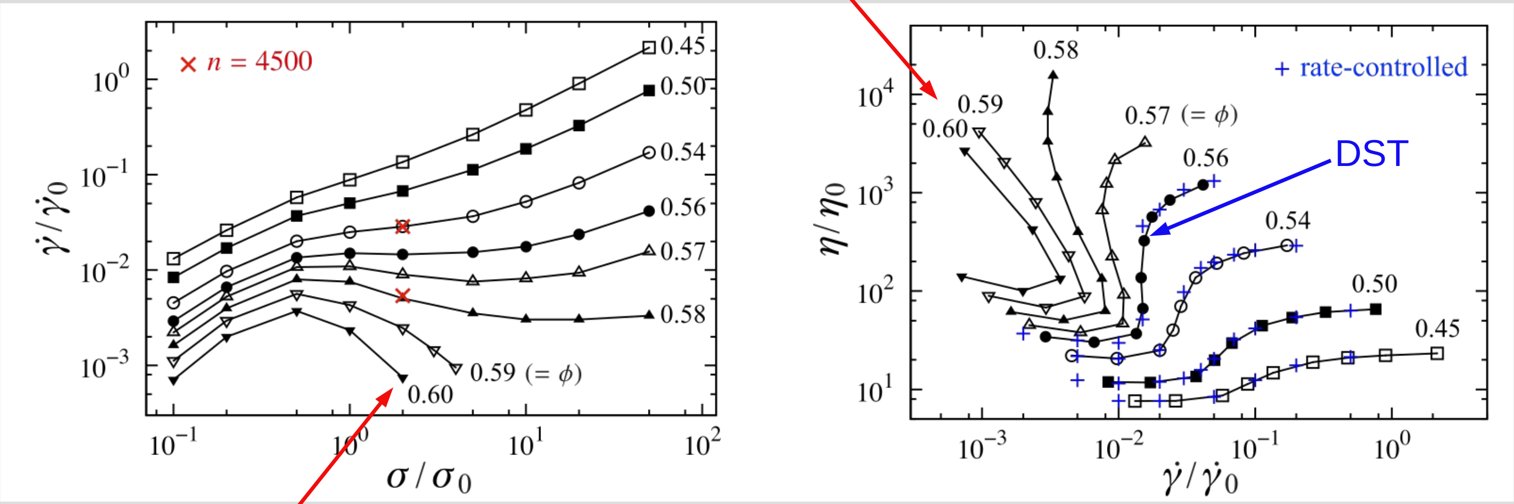



- Intermittency at DST transition (not CST)
- Intermittent contact network
- No influence of the system size

- Controlling stress is easy (using DEM w/o inertia) ⟹ Mari et al. (2015). Phys. Rev. E.
- S-shape flow curve indude DST at controlled rate ⟹ Wyart & Cates (2014). Phys. Rev. Lett.

Jamming

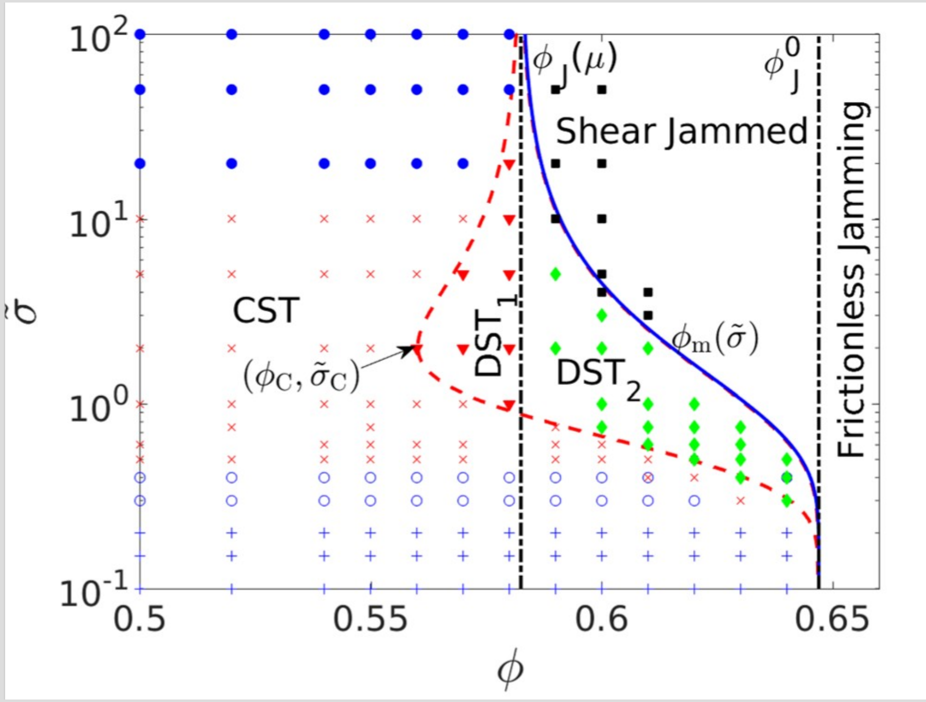


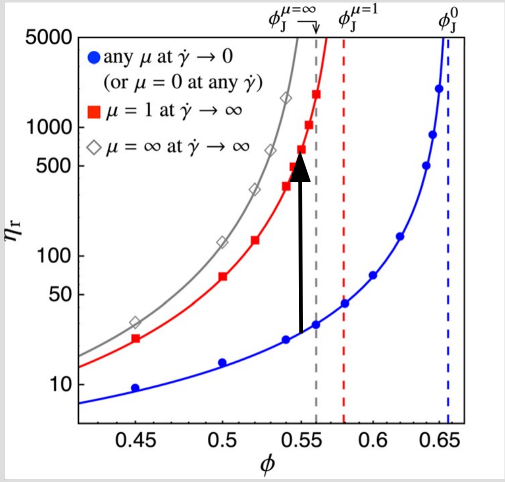

Jamming

- ➔ $\phi \leq 0.54$ rate controlled = stress controlled
- ➔ $\phi \geq 0.56$ DST in rate controlled
  S-shape in stress controlled $\dot{\gamma}(\sigma)$
- ➔ $\phi \leq 0.59$ (i.e $\phi \leq \phi_J^{mu}$) Shear jamming

- Homogeneous shear flow in the S-shape part, no phase separation
- No size effect

# Shear thickening suspensions : phase diagram




+ Thinning
- ○ Frictionless rate independent
- ● Frictional rate independent

- × CST
- ▼ DST low viscosity → high viscosity
- ◆ DST low viscosity → shear jammed
- ■ Shear jammed

# Shear thickening modelling from simulations and theory

Singh et al (2018) JOR

$$\phi_\mathsf{m}(\tilde{\sigma}) = \mathsf{f}(\tilde{\sigma})\phi_\mathsf{J}^{\mu} + [1 - \mathsf{f}(\tilde{\sigma})]\,\phi_\mathsf{J}^{0}$$

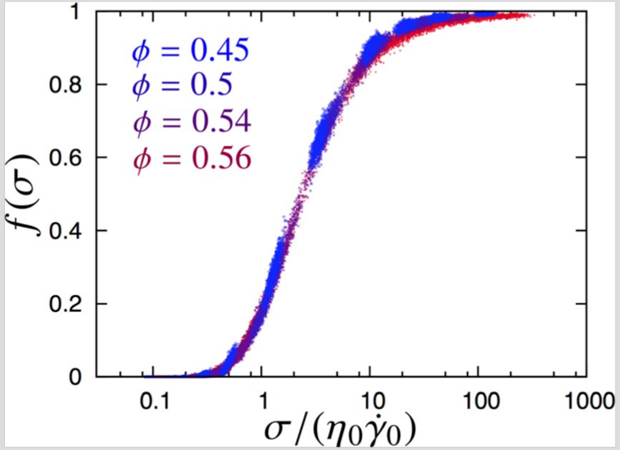
cf. Wyart & Cates (2014) PRL

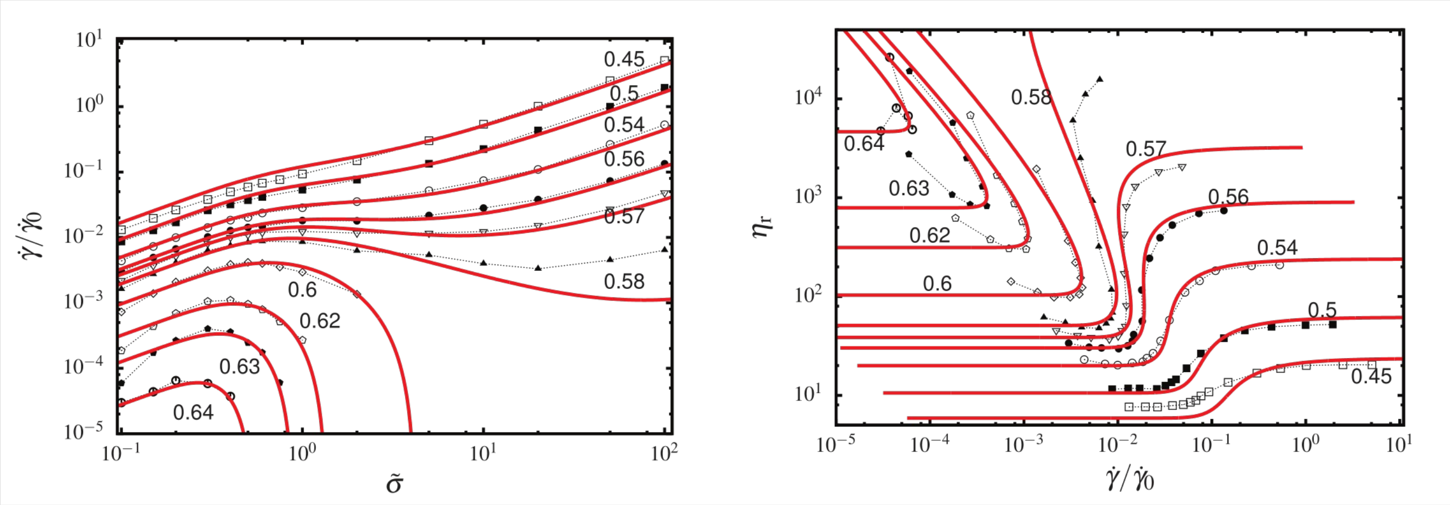
$$\alpha_\mathsf{m}(\tilde{\sigma}) = \mathsf{f}(\tilde{\sigma})\alpha^{\mu} + [1 - \mathsf{f}(\tilde{\sigma})]\,\alpha^{0}$$

$$\mathsf{f}(\tilde{\sigma}) = \exp\left(-\tilde{\sigma}^{*}/\tilde{\sigma}\right)$$

$$\boxed{\eta_\mathsf{r}(\phi, \tilde{\sigma}) = \frac{\alpha_\mathsf{m}(\tilde{\sigma})}{[\phi_\mathsf{m}(\tilde{\sigma}) - \phi]^2}}$$

Mari et al (2014) JOR



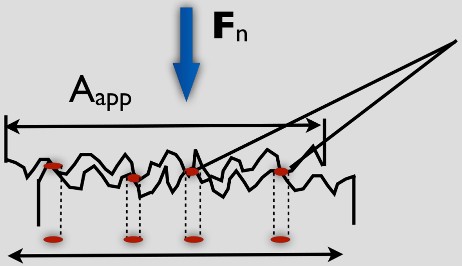




Singh et al (2018) JOR

# Shear-thinning behavior in NB suspensions

Lobry et al. (2019). Journal of Fluid Mechanics

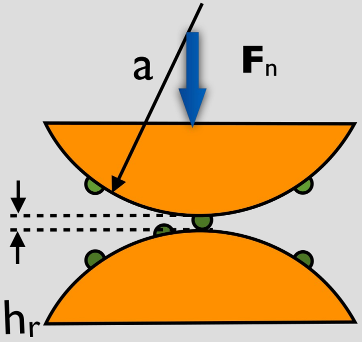
- Amontons-Coulomb law : Greenwood & Williamson (1966)




$A_r << A_{app}$

- Statistical distribution of asperity height $\longrightarrow$ $A_r \propto F_n$
- Sliding if $F_t = \sigma_s A_r \propto F_n$ $\longrightarrow$ $\boxed{\mu = \dfrac{F_t}{F_n} = constant}$

- Single asperity model : a first course




$$\boxed{F_c \propto \frac{Y^3}{E^2} {R_r}^2}$$

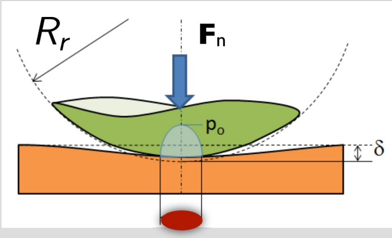
E : Young modulus

Y : yield strength

- Elastic regime ( $F_n < F_c$) :

Hertz : $A \propto {F_n}^{\frac{2}{3}}$ $\qquad F_t = \sigma_s A \propto {F_n}^{\frac{2}{3}}$

$$\boxed{\mu = \frac{F_t}{F_n} \propto {F_n}^{-\frac{1}{3}}}$$

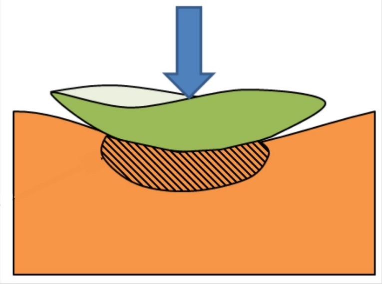



A

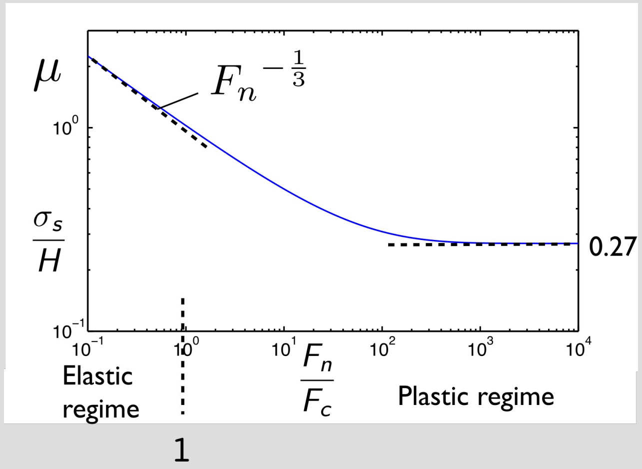
- Plastic regime ( $F_n > F_c$) :

$$\bar{p} = \frac{F_n}{A} \longrightarrow \mathrm{H} \Rightarrow \boxed{\mu = \frac{\sigma_s}{H} = constant}$$

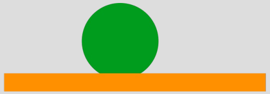
H = hardness



# Brizmer model of elasto-plastic contact




## Elastic-plastic contact model



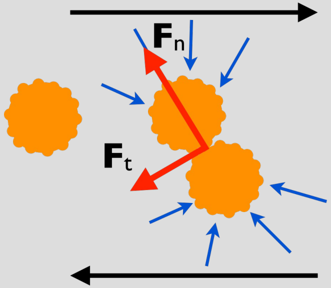
FEM simulations (Brizmer et al. Tribology Letters 2007)

$$\boxed{\mu = 0.27 coth\left[0.27\left(\frac{F_n}{F_c}\right)^{0.35}\right]}$$

$$F_c \propto \frac{Y^3}{E^2} R_r^{\,2}$$

$F_c$ : force scale for the variation of μ

## Stress scale




$$F_n \sim 6\pi a^2 \Sigma \rightarrow \frac{F_n}{F_c} \sim \frac{6\pi a^2 \Sigma}{F_c} = \frac{\Sigma}{\Sigma_c}$$

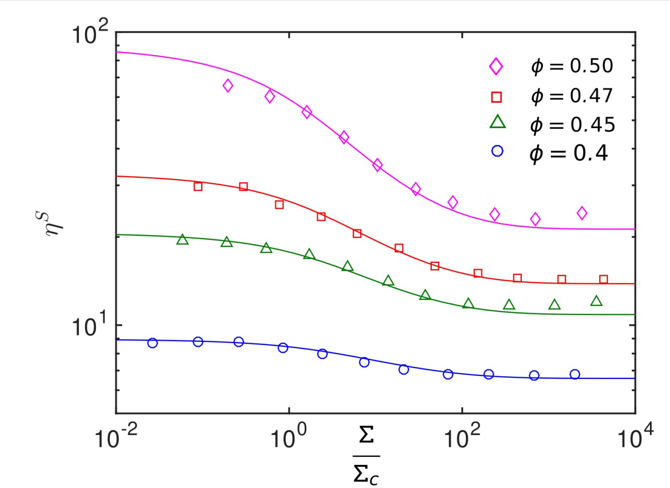
$$\Sigma_c = \frac{F_c a}{6\pi a^2}$$

# Simulations and model

$$\frac{\Sigma}{\Sigma_c} \longrightarrow 0 \quad \mu \longrightarrow \infty$$

$$\eta^s \longrightarrow \eta^s_0 = \frac{\alpha_0(\mu=\infty)}{(1-\frac{\phi}{\phi_m(\mu=\infty)})^2}$$

$$\Phi_m(\mu=\infty) \approx 0.545$$




$$\frac{\Sigma}{\Sigma_c} \longrightarrow \infty \quad \mu \longrightarrow 0.27$$

$$\eta^s \longrightarrow \eta^s_\infty = \frac{\alpha(\mu=0.27)}{(1-\frac{\phi}{\phi_m(\mu=0.27)})^2}$$

$$\Phi_m(\mu=0.27) \approx 0.622$$

Simple model $\quad \frac{F_n}{F_c} \sim \frac{\Sigma}{\Sigma_c}$

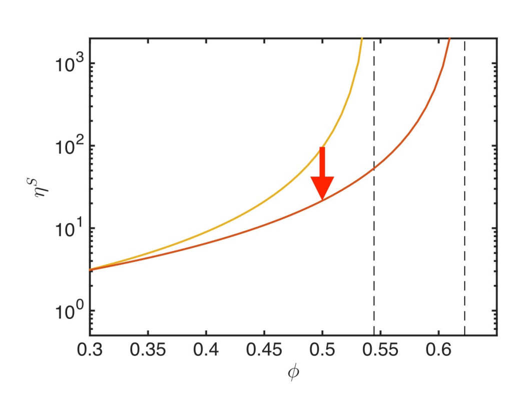
$$\mu^{eff} = 0.27 coth\left[0.27\left(\alpha_{fit}\frac{\Sigma}{\Sigma_c}\right)^{0.35}\right] \approx <\mu>$$

$$\eta^s\left(\frac{\Sigma}{\Sigma_c},\phi\right) = \frac{\alpha_0(\mu^{eff})}{\left[1-\frac{\phi}{\phi_m(\mu^{eff})}\right]^2}$$

$\alpha_0(\mu), \phi_m(\mu) \leftarrow$ Simulations at constant μ

$\alpha_{fit} = 0.59$

$$\frac{\bar{F}_n}{F_c} = 0.59\frac{\Sigma}{\Sigma_c}$$




μ from experiments : Arshad, et al. (2021). Soft Matter

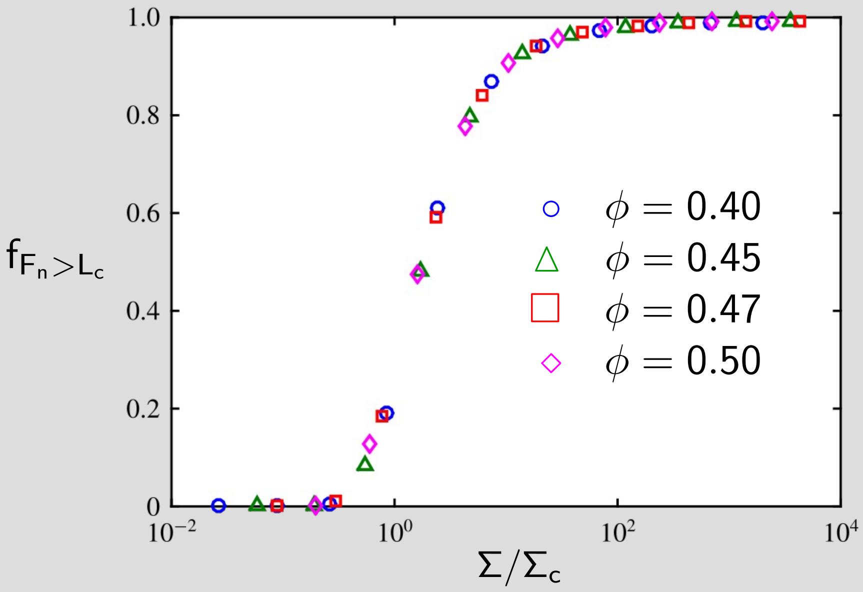
# Microscopic measurements

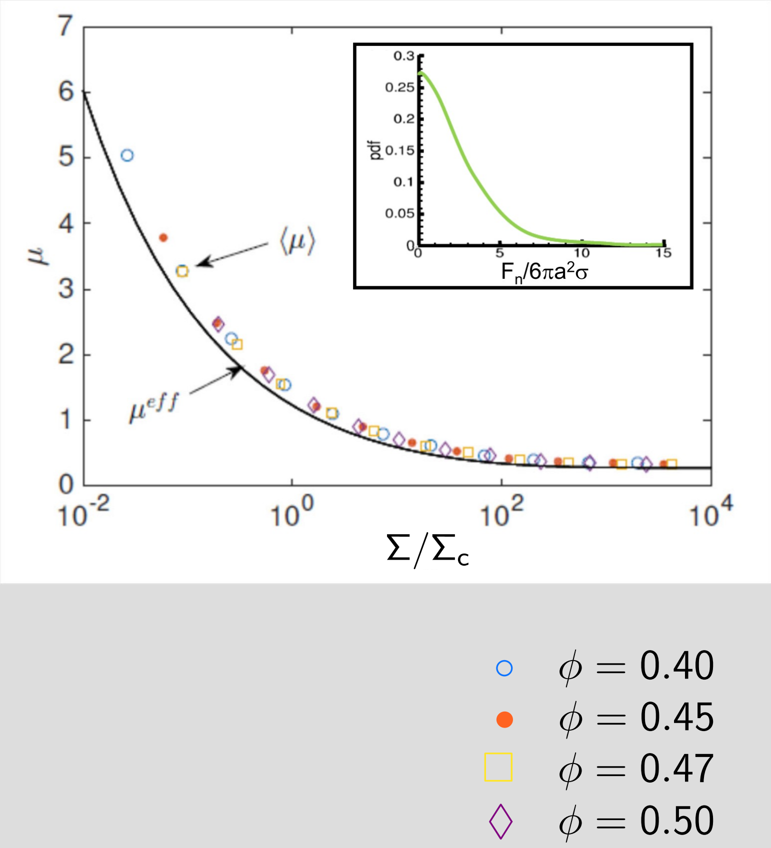
- Mean friction coefficient

Model | Simulations

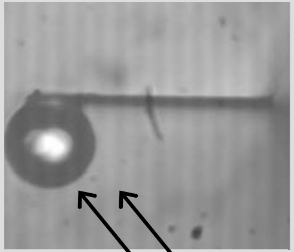
$$\mu^{eff} = 0.27 coth\left[0.27\left({\color{red}a_{fit}}\frac{\Sigma}{\Sigma_c}\right)^{0.35}\right] \approx \langle\mu\rangle_{\text{contacting particles}}$$

- Fraction of plastic contacts

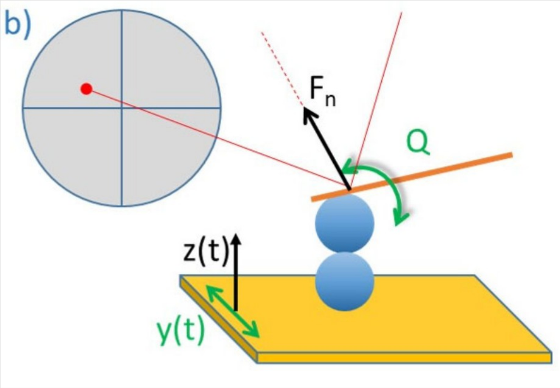


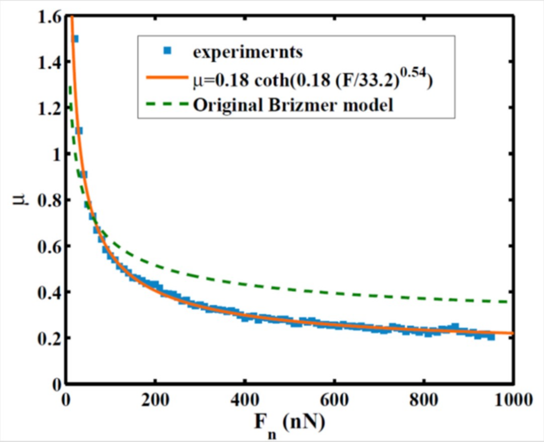
# Experiments Arshad et al. (2021) Soft Matter.

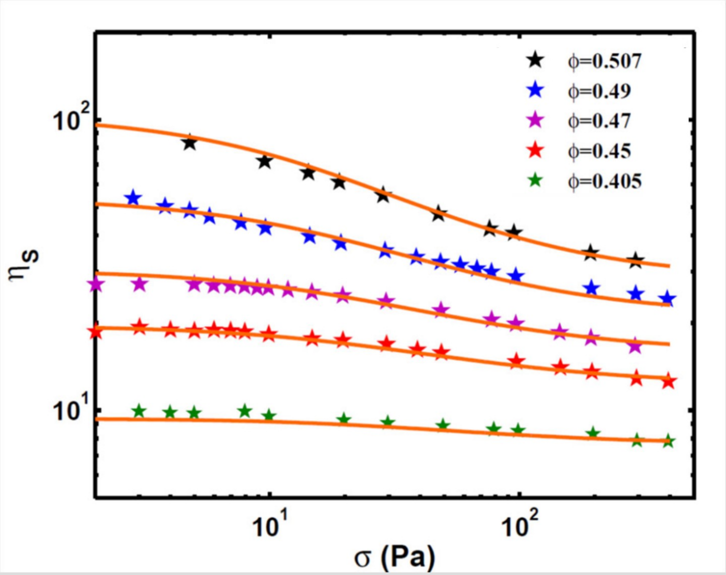


Polystyrene particle 2a=40μm
(Dynoseeds TS40)




Liquid : water+ Ucon oil




$\left.\begin{array}{c}\mu(F_N) \text{ or } \mu(\sigma) \\ + \\ \eta_S(\mu)\end{array}\right\} \longrightarrow \eta_S(\sigma)$

*Lobry et al. JFM 2019*

# Conclusion

- Not enough time to talk about :
  - Heterogeneous flows (migration / sedimentation, resuspension)
  - Other interactions  (adhesion, rolling friction...).

- Primary interest : microscopic measurements (AFM, SFA) for relevant modelling

- Beyond rigid particles ? Very soft particles, elasto-hydrodynamics ...

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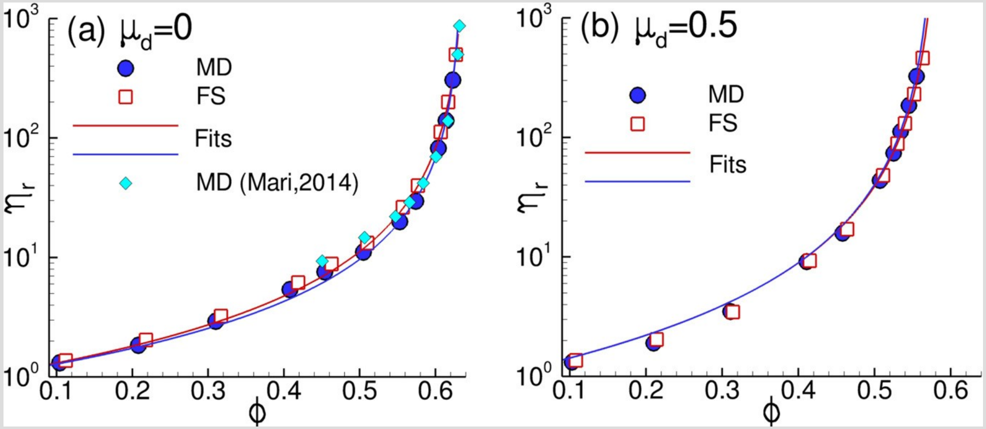
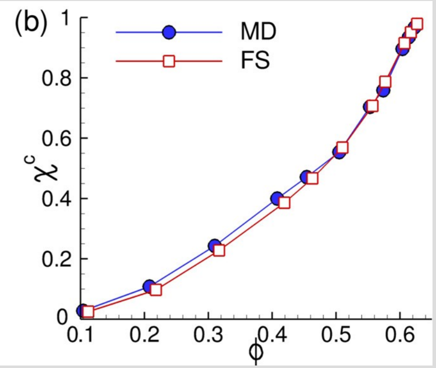
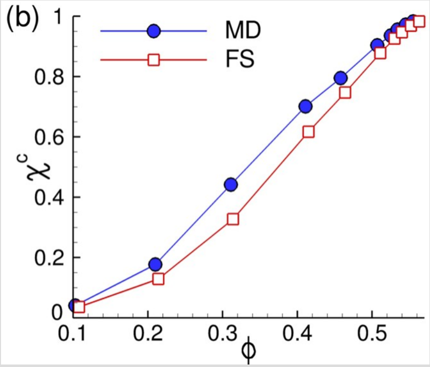
# Supplementary

# Long range hydrodynamic interactions ?

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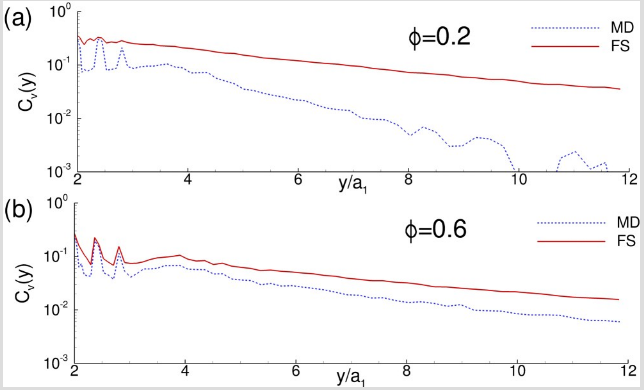


# Moments de la contrainte

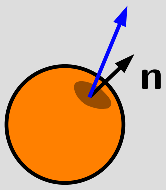
Force totale sur la sphère

$$\vec{F} = \oint\!\!\oint_{(\partial\Omega)} \bar{\bar{\sigma}}.\vec{n}\, dS = 0$$

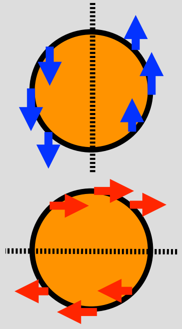
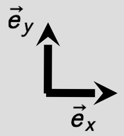
$$\Sigma = \boldsymbol{\sigma} \cdot \mathbf{n} dS$$

Couple total sur la sphère

$$\vec{T} = \oint\!\!\oint_{(\partial\Omega)} \vec{r} \times \bar{\bar{\sigma}}.\vec{n}\, dS = 0$$




$$T_z = \oint\!\!\oint_{(\partial\Omega)} (\Sigma_y x - \Sigma_x y)\, dS = 0 \qquad T_z = D_{yx} - D_{xy}$$


$$D_{yx} = \oint\!\!\oint_{(\partial\Omega)} \Sigma_y x\, dS \neq 0$$

$$D_{xy} = \oint\!\!\oint_{(\partial\Omega)} \Sigma_x y\, dS \neq 0$$

**$D_{xy}$ et $D_{yz}$ considérés séparément : efficacité de la distribution de contrainte pour cisailler la particule**

Moment d'ordre 2 de la distribution de contrainte = moment **dipolaire** (de force)

$$\boxed{D_{ij} = \oint\!\!\oint_{(\partial\Omega)} \sigma_{ik} n_k\, x_j\, dS \qquad \bar{\bar{D}} = \oint\!\!\oint_{(\partial\Omega)} (\bar{\bar{\sigma}}.\vec{n})\, \vec{x}\, dS}$$

9 composantes

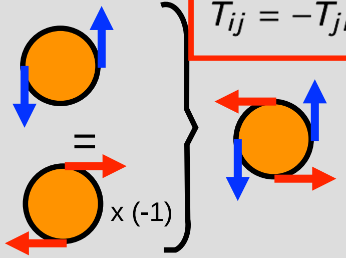
$$\boxed{\bar{\bar{D}} = \bar{\bar{S}} + \bar{\bar{T}}}$$

$$\boxed{T_{ij} = \frac{1}{2} \oint\!\!\oint_{(\partial\Omega)} [\sigma_{ik} n_k\, x_j - \sigma_{jk} n_k\, x_i]\, dS \qquad \bar{\bar{T}} = \frac{1}{2} \oint\!\!\oint_{(\partial\Omega)} [\, (\bar{\bar{\sigma}}.\vec{n})\, \vec{x} - \vec{x}\, (\bar{\bar{\sigma}}.\vec{n})\, ]\, dS}$$

Rotlet

Partie anti-symétrique $\Longleftrightarrow$ couple sur la particule

$$\boxed{T_{ij} = -T_{ji}}$$




$$\boxed{S_{ij} = \frac{1}{2} \oint\!\!\oint_{(\partial\Omega)} [\sigma_{ik} n_k\, x_j + \sigma_{jk} n_k\, x_i]\, dS \qquad \bar{\bar{S}} = \frac{1}{2} \oint\!\!\oint_{(\partial\Omega)} [\, (\bar{\bar{\sigma}}.\vec{n})\, \vec{x} + \vec{x}\, (\bar{\bar{\sigma}}.\vec{n})\, ]\, dS}$$

Stresslet

Partie symétrique

$$\boxed{S_{ij} = S_{ji}}$$